Chapter 7
Inanimate Bodies Start Moving by Themselves

Abstract This chapter concerns the XIX century. In the first section of the chapter Coriolis' studies on relative motion and Hamilton's formulation of the laws of mechanics are presented. The second section deals with applied mechanics, a discipline cultivated by the emerging figure of the scientific engineer. The case study of undershot waterwheels has been given a quite large space. The contributions of physicists and engineers such as Poncelet, Poisson and Coriolis are presented. A hint is also given to the theory of elasticity that became an essential tool in the design of machines. The chapter ends with the discussion of thermodynamical issues and the fundaments of thermal machines.

7.1 Achievements and People

In this period the field of mathematical astronomy reached a great maturity. An example of that is the discovery of the planet Neptune. Unusual changes in the orbit of Uranus led astronomers to postulate the existence of another planet, since nobody doubted Newton's law of universal gravitation. Its orbit was calculated and, pointing the telescope toward the theoretical position with a minimum gap, the planet was identified in 1846. Laplace gave an affirmative answer to the question of whether the solar system was stable. For terrestrial bodies the motion of deformable bodies started to be studied; not only the oscillations of a rope but also those of plates and three-dimensional bodies and the propagation of waves in solids was studied also. But perhaps the main focus, at least in the first half of the XIX century, was the motion of bodies by means of machines, which were no longer those of Heronian memory, but rather complex systems suitable to transmit and amplify forces and motions from one point to another. The novelty is that these machines could move by themselves, not only by the action of air and wind, which was easy to accept, but also by means of heat, whose action was instead seen as decidedly mysterious. Some relevant scientists of the period are:

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- André Marie Ampère (1775–1836). French mathematician and physicist, the founder of electrodynamics. He was involved also in works on mechanics and philosophy of science.
- Jean Charles de Borda (1733–1799). French mathematician, physicist, engineer. Worked on fluid mechanics and studied fluid flow in many different situations.
- Augustin Louis Cauchy (1789–1857). French mathematician and engineer who was an early pioneer of analysis. Produced important works in the field of mathematical physics, in particular in the theory of deformable continuous bodies.
- Gaspard Gustave Coriolis (1792–1843). French mathematician and physicist who discovered the effect bearing his name. Coriolis established the use of 'work' as a technical term in mechanics.
- Charles Augustin Coulomb (1736–1806). French mathematician, physicist and engineer. In electricity he found the law that has his name. In mechanics he developed approaches taking account of friction. In particular theories for arches and for earth slopes.
- Pierre Maurice Marie Duhem (1861–1916). French physicist, mathematician, historian and philosopher of science. He made major contributions to the science of his day, particularly in the field of thermodynamics.
- William Rowan Hamilton (1805–1865). Irish mathematical and physicist. Reshaped the theoretical optics and mechanics by basing them on his law of varying action. In mathematics he introduced the concept of quaternions.
- Hermann Helmholtz (1821–1894). German scientist; one of the few scholars to master two disciplines: medicine and physics. In 1847 Helmholtz wrote the famous *Uber die Erhaltung der Kraft*.
- Heinrich Rudolf Hertz (1857–1894). German physicist, whose experiments were fundamental for the discovery of the radio. He was the supporter of a mechanics without forces, based on hidden masses.
- James Prescott Joule (1818–1889). English physicist, proved that mechanical, electric, thermal and other form of energy are interconvertible on a fixed basis. He was also a prolific inventor.
- Gustav Robert Kirchhoff (1824–1887). German physicist and mathematician. Known especially for his work with the spectroscope. He also carried out important researches in electricity thermodynamics and continuum mechanics.
- Gabriel Lamé (1795–1870). French mathematician and engineer. His work on differential geometry and contributions to Fermat's last theorem are important. Relevant his studies on the theory of elasticity.
- Pierre Simon Laplace (1749–1827). French mathematician and physicist. The champion of the corpuscular mechanics, on which most of French scientists adhered.
- James Clerk Maxwell (1831–1879). Scottish mathematician and physicist. His most prominent achievement was the formulation of a set of equations that describe the electromagnetic field. He also contributed to mechanics and the theory of elasticity.
- Julius Robert Mayer (1814–1878). German physician and physicist. Best known for enunciating in 1841 one of the original statements of the conservation of energy.

- Gaspard Monge (1746–1818). French mathematician, the inventor of descriptive geometry and one of the fathers of differential geometry.
- Claude Louis Navier (1785–1836). French engineer and physicist was the prototype of the *engineer savant*. He was known for hist studies on the theory of elasticity and hydrodynamics.
- Friedrich Wilhelm Ostwald (1853–1932). Baltic German chemist, Nobel prize in chemistry in 1909. Very productive in an extremely broad range of fields. He his well known as the prophet of energetism.
- Jules Henri Poincaré (1854–1912). French mathematician, physicist and a philosopher of science. He made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He is also considered to be one of the founders of the field of topology.
- Siméon Denis Poisson (1781–1840). French mathematician and physicist. He was the promotor of the *mécanique physique*. The Poisson distribution in probability theory is named after him. His treatise of mechanics had a great diffusion through Europe.
- Jean Victor Poncelet (1788–1867). French engineer and mathematician, considered as the founder of modern projective geometry. As a professor of mechanics he improved the design of turbines and waterwheels.
- Venant Adhémar Jean Claude Barré (1797–1886). French mathematician and engineer. He got his main results in mechanics of continuum and theory of elasticity. He anticipated the vector calculus and was the more fierce supporter of the molecular theory of matter.
- William Thomson (Lord Kelvin) (1824–1907). Irish mathematical physicist and engineer. He did important work in the mathematical analysis, in the electricity and in the formulation of the first and second laws of thermodynamics.
- James Watt (1736–1819). Scottish mechanical engineer whose improvements to Newcomen's steam engine were fundamental to the changes brought by the Industrial Revolution.

7.2 The Framework

The period at the turn of the XIX century was characterized by profound political, social, technological, scientific changes. It was a period that saw the affirmation of the bourgeoisie and the emergence of the modern bureaucratic State. Even science and technology suffered major changes. Not for nothing the XIX century was the century in which the term scientist was coined. Scientists found support in the state; support that resulted in the creation of scientific and technical schools. They found support even in private investment, particularly among middle-class owners of the nascent manufacturing industry.

Science ceased to be the business of an elite who practiced it for pleasure, to become a social activity. It was realized that the study of technical problems could not be left in the hands of practical people but must be taken by savants who knew

the development of science and/or by scientists themselves. The birth of the École polytechnique, strongly supported by the ruling class in the revolutionary France, was the symbol of the desire to connect science with technology.

The second half of the XVIII century saw the emergence in Britain of what is commonly called the first Industrial Revolution, or the Industrial Revolution tout court. The fundamental sectors of the Industrial Revolution were the manufacture of cotton and steel metallurgy. While the manufacture of cotton was a case in point, where it clearly appeared how technological innovation could radically increase productivity and therefore reduce the price, the steel metallurgy was a driving case also, as the developments internal to the organization made possible the developments in other sectors. The development of the steel industry was made possible by the use of coal in the blast furnace and the use of steam, both in furnaces for insufflation of the heat and in the forging, for example to provide energy to hammers, or in laminating.

The steam engine, at least in the collective imagination, is the icon of the industrial revolution. It allowed industries everywhere to automate repetitive and heavy tasks, even where there were no natural sources for water or wind power. The steam engine was also instrumental in the transport sector. In 1825 the first railway of the history was inaugurated—between Stockton and Darlington—and in 1830 the first line between two major cities, Liverpool and Manchester was inaugurated. Then the railroad as well as the industrial revolution spread into the continent.

As far as science is concerned a new and important phenomenon was the division and specialization of knowledge. First the fundamental division between science and philosophy, then the division within science, between mathematics and experimental sciences and within them. At the beginning of the century there were people like Pierre Simon Laplace (1749–1827) and Alexander Von Humboldt (1769–1859) who had a unified vision of science, but they were isolated cases, the last offshoot of the Enlightenment. Within the empirical sciences the division into branches such as chemistry, electromagnetism, thermodynamics came from the inability to coordinate the large amount of new laws and to explain new phenomena, especially in the light of the efficiency that was required of science in its relations with technology. Within mathematics the division was still due to the difficulty of coordinating the various branches, not only for the extent that they have achieved, but also for the need for rigor which was a novelty of the XIX century which involved a different (axiomatic) foundation of the various sectors: mathematics, projective geometry, algebra, etc., which created barriers.

The efforts of XVIII century for the development of a unified theory of the motion of bodies, which ideally can be considered found their end with Lagrange's 1788 *Méchanique analitique*, had given classical mechanics a modern aspect. Discussion upon principles passed now from their kind to their justification and their utility.

The issue of Lagrange's treatise *Méchanique analitique* of 1788 and its great success, due mainly to the use of the principle of virtual work, was the occasion in the XIX century for a deep discussion on the logical status of the principles of mechanics, mainly statics, probably still more heated than that of half of the XVIII century on the principles of dynamics. However, after this debate, the science of

motion found itself at a turning point. The models of mass point and rigid body, used by the mechanicians of the XVIII century, had now exhausted their outcomes: the problems which could be solved with them are either too difficult, such as the problem of n bodies, or of little importance. The turning point was reached in two different directions:

1. Perfecting the theoretical aspects. Studying phenomena not yet explored of the classical mechanics, as those observed in moving frames, and the adoption of a more power formal language for the theory.
2. The opening of new perspectives, as the introduction of the deformation of bodies, either adopting a molecular or a continuum model for matter and introduction of the concepts of energy and dissipation of thermodynamics.

This Hamilton's comment deserves attention:

> But the science of force, or of power acting by law in space and time, has undergone already another revolution, and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginably conditions, which Lagrange so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods. And the method proposed in the present essay, for the deductive study of the motions of attracting or repelling systems, will perhaps be received with indulgence, as an attempt to assist in carrying forward so high an inquiry [175].[1]

7.2.1 The Naturphilosophie

Besides a general appreciation of the development of the science of motion, as well of physics in general, according to the lines suggested by Newton, Euler, Lagrange, based on experiments and mathematics, there was also resistance by many philosophers and some scientists. Hereinafter one case of these contrasting movements is referred to, that of the Naturphilosophie.

Naturphilosophie is a term used by English speakers to identify a current in the philosophical German tradition in the earlier XIX century as applied to the study of nature. German speakers use the expression *Romantische Naturphilosophie*, the philosophy of nature developed at the time of Romanticism [637, 604]. Naturphilosophie greatly influenced scientific thought at the turn of the XIX century in Germany and even elsewhere. Some historians see a surge of nationalism at the basis of this movement. Indeed after the Napoleonic conquer of Germany it would have been natural for the defeated people to react to the ideology represented by French men, the Enlightenment and the natural science founded on experiments and mathematics. The Naturphilosophie replaced the mechanical-atomist explanation of nature by

[1] p. 247.

dynamic-organic concepts, with the substitution, sometimes, of sentiment to critical reason, of intuition to experiment. Of course there was a gradation; on the one hand the idealist philosophers Fichte, Schelling and Hegel. On the other hand some great scientists, who retained only some aspects of the Naturphilosophie, such as for example the unity of nature.

In the foundation of the Naturphilosophie there are some ideas expressed by Immanuel Kant (1724–1804) in his *Metaphysishe Anfangsgrnde der Naturwissenschaft* of 1786 [197]. According to Kant, the actual reality of matter was in contrast to the mechanist conception of nature: matter indeed shows two original forces, of attraction and repulsion, which in turn cause motions. Matter does not fill space by virtue of its absolute impenetrability but by means of a repulsive force that has its degree of intensity that is different in different forms of matter. Of the two forces Kant considered the repulsion force as the fundamental one.

Reading Kant's *Kritik der Urteilskraft* (1790), alongside the *Metaphysische Anfangsgrnde der Naturwissenschaft*, would have been decisive for many of the protagonists of the romantic science. In Kant's theses many questions about the unity of the sciences and the value of their categorical apparatus were raised in which the array of many aspects of romantic reflection on knowledge of nature founded their source. For Johann Wolfgang Goethe (1749–1832) the reading of Kant would have been of great importance. In a specific area of scientific inquiry, Goethe's vision of nature exerted a decisive influence on the romantic science debate: it was the optics and especially, what it is so to speak, the destruens pars of Newton's theory of color, coming to produce intense and significant reactions in the entire German scientific debate of the early decades of the century.

Among the philosophers who developed Kant's ideas, Friedrich Wilhelm Joseph Shelling (1775–1854) was the one who developed the greatest interest on natural sciences. Nature and mind were identical to Shelling; and all life could only be understood when nature is assumed to have infinite activity. Shelling, as all the Natur-philosophers (and Natur-scientists also) held that everything in nature is due to polar interaction of opposites, as for instance mind and matter, the pole of electricity or of magnetism. Among the scientists to signal the conscious adhesion to the Naturphilosophie were great scientists such as Johann Wilhelm Ritter (1766–1810) and Hans Christian Oersted (1777–1851). One of the positive effect on Ritter of the Naturphilosophie was the prediction of the ultra-violet light, as a reaction to Frederick William Herschel's (1738–1822) ideas. Richter, based on the polarity concept maintained that there should exist light in the opposite side of the spectrum with respect to the infra-red one. However Ritter was primarily an experimenter and not a Nature-philosopher *strictu sensu*. The same holds good for Oersted (Østerd) who although Danish was largely influenced by German culture. At the beginning of his career he was a Shelling's follower, with some criticism however. A major influence of the Naturphilosophie on Oersted was the discovery of the action of electric current on a magnetic field, a direct consequence of his belief in the unity of all natural forces. One more important scientists to be cited is Henrik Steffens (1773–1845), a geologist.

Of course there was opposition to the Naturphilosophie by many German scientists; but they were less apt to engage in public disputes. For example Johann

Carl Friedrich Gauss (1777–1855) was highly opposed to romantic speculation in mathematics. The greatest opposition came however from the experimental scientists; especially important was the role of Ludwig Wilhelm Gilbert (1769–1824), editor of the prestigious Annalen der Physik, who was followed by Johann Christian Poggendorff (1796–1877) in the new version of the journal Annalen der Physik und Chemie.

Naturphilosophie was in fact on the wane by the 1840s because of its sterility. However its influx still persisted in many German scientists especially in biology. For instance Helmholtz and Mayer were surely influenced by the Naturphilosophie in conceiving the principle of conservation of energy, which was quite a corollary of the principle of unity of the forces of nature professed by the Naturphilosophie.

7.3 Perfecting the Theoretical Aspects

In this section I treat two themes that are very different from each other both related with the internal completion of the science of motion: the theory of relative motion developed by Gustave Gaspard Coriolis and the analytical mechanics developed by William Rowan Hamilton (1805–1865) and Carl Gustav Jacob Jacobi (1804–1851). The first theme is very simple to study from a technical point of view and in the modern courses in rational mechanics it is relegated to kinematics and carried out in a few steps by means of the vector calculus. It is however historically very important because it represented a theoretical completion of mechanics and, at the beginning, it was surely seen as a dynamical and not merely as a kinematical problem. The second theme requires a quite complex, even for the modern standard, mathematical apparatus; it opened the way to the rational mechanics of the XX century, characterized by the introduction of a general formulation and a renewed form of geometrical and algebraic language; for example the vector algebra—widely known at the end of the XIX century—turned out to be a tremendous tool of rationalisation.

7.3.1 The Study of Relative Motions

Coriolis considered the problem of finding the laws of motion in a frame moving in any way with respect to a fixed frame. He did this within the scope of his efforts to understand some aspects of machines, in particular of hydraulic machines, stimulated by Poncelet's results on overshot waterwheels, using the principle of living forces:

> The principle of living forces extended to the relative motions gives very easily an exact theory of the waterwheels like that of de Borda or the turbines of Mr. Burdin.[2] For the wheels with curved paddles of Poncelet, it shows that whenever the water comes down from

[2] Claude Burdin, (1788–1873) was the inventor of the first turbine which was then perfected by his pupil Benoît Fourneyron.

> the paddle at same distance from the axis of rotation where it came in, if one neglects friction, it cannot have gained or lost but the relative speed due to the action of gravity relative to the wheel considered at rest; so that, according to the usual form of the paddles, the relative velocity of water is greater when leaving than when entering [95].[3] (A.7.1)

Coriolis treated the problem in two memoirs: *Sur le principe des forces vives dans le mouvements relatifs des machines* of 1831 and *Sur les équations du mouvement relatif des systèmes de corps* of 1835, both published in the Journal de l'École polytechnique [95, 96].

7.3.1.1 Coriolis' First Theorem

In Coriolis' first memoir, *Sur le principe des forces vives dans le mouvements relatifs des machines* of 1831, the problem was that of the motion of a system of mass points in a reference frame (x, y, z) moving with respect to a fixed one (x_1, y_1, z_1). The moving frame is individuated by the coordinate ξ, η, ζ of its origin and by the director cosines (a, b, c) of x axes, (a', b', c') of y axes, (a'', b'', c'') of z axes, with respect to (x_1, y_1, z_1). The study started from the motion of a single mass point of mass m, constrained by equations of the kind $L = 0$, $M = 0$ depending on the coordinates of the moving frame, so that the components of the constraint reactions with respect to (x, y, z) are given by [95][4]:

$$\lambda \frac{dL}{dx} + \mu \frac{dM}{dx} + \&\text{c.}, \quad \lambda \frac{dL}{dy} + \mu \frac{dM}{dy} + \&\text{c.}, \quad \lambda \frac{dL}{dz} + \mu \frac{dM}{dz} + \&\text{c.} \tag{7.1}$$

λ, μ, etc. being arbitrary coefficients

By indicating with X_1, Y_1, Z_1 the components of the forces acting on the mass point in the fixed frame, the following equations of motion can be obtained with a change of coordinates from (x, y, z) to (x_1, y_1, z_1) [95][5]:

$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= X_1 + a\lambda\frac{dL}{dx} + b\lambda\frac{dL}{dy} + c\lambda\frac{dL}{dz} + \&\text{c.}, \\ m\frac{d^2y_1}{dt^2} &= Y_1 + a'\lambda\frac{dL}{dx} + b'\lambda\frac{dL}{dy} + c'\lambda\frac{dL}{dz} + \&\text{c.}, \\ m\frac{d^2z_1}{dt^2} &= Z_1 + a''\lambda\frac{dL}{dx} + b''\lambda\frac{dL}{dy} + c''\lambda\frac{dL}{dz} + \&\text{c.} \end{aligned} \tag{7.2}$$

[3] p. 271. My translation.

[4] p. 273.

[5] p. 273.

With a new change of coordinates, now from x_1, y_1, z_1 to x, y, z, after some mathematics, Coriolis obtained [95][6]:

$$\begin{aligned}
&md^2x + 2mdy(adb + a'db' + a''db'') + 2mdz(adc + a'dc' + a''dc'') + X_e \\
&\quad = X + \lambda\frac{dL}{dx} + \&\text{c.}, \\
&md^2y + 2mdx(bda + b'da' + b''da'') + 2mdz(bdc + b'dc' + b''dc) + Y_e \\
&\quad = Y + \lambda\frac{dL}{dy} + \&\text{c.}, \\
&md^2z + 2mdx(cda + c'da' + c''da'') + 2mdy(cdb + c'db' + c''db'') + Z_e \\
&\quad = Z + \lambda\frac{dL}{dz} + \&\text{c.}
\end{aligned} \tag{7.3}$$

where all the quantities, force included, are referred to the moving frame (x, y, z). For the 'sake of economy' Coriolis had avoided writing the differential dt, so that for example d^2x means d^2x/dt^2, da means da/dt, etc. The quantities X_e, Y_e, Z_e are the components in (x, y, z) of the dragging forces, the forces necessary to maintain at rest the mass point if no external forces were acting, with sign reversed.

To obtain the principle of living forces from Eq. (7.3) Coriolis multiplied the first equation for dx (the velocity with respect to x), the second for dy, the third for dz, assumed compatible with the constraints $L = 0$, $M = 0$, &c., and summing over all the mass points. As the reactive forces vanish, being orthogonal to the compatible velocities, the following equations were obtained [95][7]:

$$\begin{aligned}
&\sum m(dxd^2x + dyd^2y + dzd^2z) \\
&\qquad + \sum 2mdxdy(adb + a'db' + a''db'' + bda + b'da' + b''db'') \\
&\qquad + \sum 2mdxdy(cda + c'da' + c''da'' + adc + a'dc' + a''dc'') \\
&\qquad + \sum 2mdxdy(bdc + b'dc' + b''dc'' + cdb + c'db' + c''db'') \\
&\qquad + \sum (X_e dx + Y_e dy + Z_e dz) = \sum (Xdx + Ydy + Zdz),
\end{aligned} \tag{7.4}$$

that, with some mathematics, after having introduced the force P_e, whose components are opposite to X_e, Y_e, Z_e, and the force P of component X, Y, Z, and integrating gives rise to the expression[8]:

$$\begin{aligned}
\sum \frac{mV_r^2}{2} - \frac{mv_r^2}{2} &= \sum \int P\cos(Pds_r)ds_r \\
&\quad + \sum \int P_e \cos(P_e ds_r)ds_r,
\end{aligned} \tag{7.5}$$

[6] p. 275.

[7] p. 276.

[8] [95, p. 273]. Some typos are corrected in the equation.

where V_r and v_r are respectively the velocities of the mass points of the system at time t and at time 0 and ds_r their displacements in the moving frame.

From Eq. (7.5) it is clear that the principle of living force remains valid with a *slight change*, addition of the work of dragging forces:

> This equation contains the theorem for which the principle of forces still takes place in the motion relative to movable axes, provided that, to the action quantities $\int Pcos(Pds_r)ds_r$ evaluated with the given forces P arcs ds_r described in this relative motion, other quantities of action are added which result from the forces P_e, equal and opposite to those that one should apply to each mobile point to make it to take the motion it would have if it were invariably linked to the moving axes [95].[9] (A.7.2)

This is what René Dugas calls *Coriolis' first theorem* [450].[10]

7.3.1.2 Coriolis' Second Theorem

The second Coriolis memoir *Sur les équations du mouvement relatif des systèmes de corps* of 1835 is perhaps more interesting because it made explicit the forces, that, though they do not intervene to modify the expression of living forces, however contribute to motion. These forces, which Coriolis named *compounded centrifugal forces* (forces centrifuges composées), are now known as *Coriolis forces*.

Coriolis started from Eq. (7.3) of the 1831 memoir, which by introducing the three components p, q, r of the angular velocity of the moving frame with respect to the fixed one, referred however to (x, y, z), with passages that Coriolis did not write, give [97]:[11]

$$\begin{aligned}
\frac{d^2x}{dt^2} &= 2\left(rm\frac{dy}{dt} - qm\frac{dz}{dt}\right) + X - X_e + \lambda\frac{dL}{dx} + \&\text{c.},\\
\frac{d^2y}{dt^2} &= 2\left(pm\frac{dz}{dt} - rm\frac{dx}{dt}\right) + Y - Y_e + \lambda\frac{dL}{dy} + \&\text{c.},\\
\frac{d^2z}{dt^2} &= 2\left(qm\frac{dx}{dt} - pm\frac{dy}{dt}\right) + Z - Z_e + \lambda\frac{dL}{dz} + \&\text{c.}
\end{aligned} \tag{7.6}$$

Coriolis noted that besides the dragging forces X_e, Y_e, Z_e there are also other forces having components in the moving reference [97]:[12]

$$2\left(rm\frac{dy}{dt} - qm\frac{dz}{dt}\right), \quad 2\left(pm\frac{dz}{dt} - rm\frac{dx}{dt}\right), \quad \left(qm\frac{dx}{dt} - pm\frac{dy}{dt}\right). \tag{7.7}$$

[9] p. 277. My translation.

[10] p. 360.

[11] p. 146.

[12] p. 146.

He had some difficulty in giving a geometrical/mechanical meaning to these forces, most probably because of the lack of notions of vector calculus. He found a strong analogy with the ordinary centrifugal forces of a rotating mass point, which by indicating with ω the angular velocity of the segment tangent to motion, and with v the velocity of the mass point of mass m, can be written as:

$$\omega m v. \tag{7.8}$$

The forces given by (7.7), or better their values divided by a factor 2, have an analogous expression to (7.8), when properly considered. To this purpose ω should be reinterpreted as the vector angular velocity of the moving frame with respect to the fixed one (that is $\omega = \sqrt{p^2 + q^2 + r^2}$) and v as the component of the relative velocity of the motion of m on a plane orthogonal to the instantaneous axis of rotation of the moving frame; the vector product of ω and v gives the relations (7.7). For this reason Coriolis called the forces (7.7), divided by two, the compounded centrifugal forces.

These are Coriolis' comments on his results, which René Dugas calls *Coriolis' second theorem* [450]:[13]

> One arrives so to this property, that the expressions of the forces to add to the given forces to have the expressions of the forces in the relative motions forces are: 1^0 those opposing the forces capable of producing for each point the motion that it would have if it were bound to the mobile planes. 2^0 the double of the composed centrifugal forces [97].[14] (A.7.3)

Coriolis' memoir continued by reconnecting to the first memoir with the formulation of a theorem valid for a system of mass points, more general than the principle of living forces. If $\delta x, \delta y, \delta z$ represent a system of virtual displacements in the moving frame, that is displacement compatible with the constraints, Eq. (7.6), multiplied by $\delta x, \delta y, \delta z$ and summed, can be rewritten as [97]:[15]

$$\begin{aligned}\sum m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right) + 2p\sum m\left(\frac{dy\,\delta z - dz\,\delta y}{dt}\right)\\ + 2q\sum m\left(\frac{dz\,\delta x - dx\,\delta z}{dt}\right) + 2r\sum m\left(\frac{dx\,\delta y - dy\,\delta x}{dt}\right)\\ = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) - \sum (X_e\,\delta x + Y_e\,\delta y + Z_e\,\delta z)\,.\end{aligned} \tag{7.9}$$

This result was commented by Coriolis with a quite complex locution:

> Thus one can say that to have an equation for the relative motion one must add to the usually existing terms for absolute movement, first that which comes from the forces which are capable of forcing the points to remain constrained to the moving planes, and further a term that is equal to twice the angular velocity of the movable axes of rotation multiplied by the sum of the projections on a plane perpendicular to the axis of rotation (of the mobile planes),

[13] p. 362.
[14] pp. 147–148. My translation.
[15] p. 149.

of all the areas of the parallelograms formed by the actual amounts of motion and the virtual velocity [97].[16] (A.7.4)

From Eq. (7.9) it is evident that considering true instead of virtual displacements/velocities, so that $\delta x = dx$, $\delta y = dy$, $\delta z = dz$, the contribution of the complementary centrifugal force vanishes because for instance $dy\,\delta z - dz\,\delta y = dydz - dzdy = 0$, and after integration equation (7.5) is found again.

7.3.2 The Mechanics of William Rowan Hamilton

Mechanics was always strictly related to mathematics so much that in antiquity it was not considered distinct from it (see Chap. 1). Moreover Newtonian mechanics would have not been possible without the *Calculus*, which is involved even in the formulation of its principles. Lagrange's mechanics was the occasion to show the power of the calculus of variations; differently from Newton, Lagrange was a mathematician more interested in showing the fertility of his methods than the justification of mechanical principles. However this last point was important and the *Mécanique analytique* was considered relevant for its foundational aspects also. More difficult is to judge the role of mathematics in post-Lagrangian mechanics. Here the field of rational analytic mechanics was concerned and the formal aspects, the elegance, the clarification of the spheres of validity of the theorems had become fundamental. However the improvements that were introduced made easier the solutions of some problems and, maybe, made also possible the solution of others; for this reason it is certainly restrictive to speak of formal perfecting only.

The present section concerns the systematization of the analytical mechanics due to William Rowan Hamilton (1805–1865). It is exposed in texts that are quite difficult to be presented to non-professional mathematicians because of the amount of mathematics involved and the abstract way of exposition, which can in no way be avoided. The main scope of the section is to present the role that mathematics is assuming in the analytical mechanics that became an object of study for skilled mathematicians.

Hamilton wrote two fundamental papers on mechanics, strictly reconnected to his works on geometrical optics [450]: *On a general method in dynamics* of 1834 [176] and the *Second essay on a general method in dynamics* of 1835 [177].

In the paper of 1834 Hamilton started directly from the variational equation of motion, which was taken for granted:

$$\sum m(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \delta U, \tag{7.10}$$

where the index of the summation $\sum$, extending to all points of the system, is implicit; m is the mass of any of such points, x'', y'', z'' are their component accelerations

[16] p. 150. My translation. Coriolis devoted some space to show that products like $dy\,\delta z - dz\,\delta y$ represents areas.

in a fixed frame,[17] $\delta x, \delta y, \delta z$ are any infinitesimal arbitrary displacements which the points can be imagined to receive and δU is the infinitesimal variation of a *force function U* of the masses and the mutual distances of the several points of the system, the form of which depends on the laws of their mutual action, according to the equation:

$$U = \sum m m_{\mathsf{l}} f(r), \tag{7.11}$$

r being the distance of the two mass points m and m_{l} and the function $f(r)$ being such that the derivative $f'(r)$ expresses the law of their forces [176].[18]

Hamilton did not consider limitative the recourse to central forces only, as he assumed that this is what actually occurs in the universe.

> Professor Hamilton[19] is of opinion that the mathematical explanation of all the phenomena of matter distinct from the phenomena of life, will ultimately be found to depend on the properties of systems of attracting and repelling points. And he thinks that those who do not adopt this opinion in all its extent, must yet admit the properties of such systems to be more highly important in the present state of science, than any other part of the application of mathematics to physics. He therefore accounts it the capital problem of Dynamics 'to determine the 3 N rectangular coordinates, or other marks of position, of a free system of n attracting or repelling points as functions of the time', involving also 6 N initial constants, which depend on the initial circumstances of the motion, and involving besides, n other constants called the masses, which measure, for a standard distance, the attractive or repulsive energies [176].[20]

From Eqs. (7.10) and (7.11) the differential equations of motion are easily and usually evaluated as [176]:[21]

$$\begin{aligned}
m_1 x_1'' &= \frac{\delta U}{\delta x_1}; & m_2 x_2'' &= \frac{\delta U}{\delta x_2}; & \cdots \quad m_n x_n'' &= \frac{\delta U}{\delta x_n} \\
m_1 y_1'' &= \frac{\delta U}{\delta y_1}; & m_2 y_2'' &= \frac{\delta U}{\delta y_2}; & \cdots \quad m_n y_n'' &= \frac{\delta U}{\delta y_n} \\
m_1 z_1'' &= \frac{\delta U}{\delta x_1}; & m_2 z_2'' &= \frac{\delta U}{\delta z_2}; & \cdots \quad m_n z_n'' &= \frac{\delta U}{\delta z_n}.
\end{aligned} \tag{7.12}$$

Hamilton commented that notwithstanding the elegance and simplicity of this approach, the difficulty of solving that problem had hitherto appeared insuperable, so that "only seven intermediate integrals, or integrals, of first order" have been found.

> No general solution has been obtained assigning (as a complete solution ought to do) $3n$ relations between the n masses $m_1, m_2, \ldots m_n$, the $3n$ varying coordinates $x_1, y_1, z_1, \ldots, x_n$,

[17] Notice that, here and in the following the apex $'$ indicates the differentiation with respect to time.

[18] p. 249.

[19] Hamilton, for a rhetorical artifice, speaks of himself in third person.

[20] p. 513.

[21] p. 249.

> y_n, z_n, the varying time t, and the $6n$ initial data of the problem, namely, the initial coordinates $a_1, b_1, c_1, \ldots, a_n, b_n, c_n$, and their initial rates of increase $a'_1, b'_1, c'_1, \ldots a'_n, b'_n, c'_n$; the quantities called here initial being those which correspond to the arbitrary origin of time. It is, however, possible (as we shall see) to express these long-sought relations by the partial differential coefficients of a new central or radical function, to the search and employment of which the difficulty of mathematical dynamics becomes henceforth reduced [176].[22]

To reach the purpose expressed in the final part of the above quotation and find more suitable equations, Hamilton introduced the "whole living force $2T$":

$$T = \frac{1}{2}\sum m(x'^2 + y'^2 + z'^2), \tag{7.13}$$

from which the Eq. (7.10), which still holds by replacing the variation δ with the differential d, gives:

$$dT = dU \tag{7.14}$$

and by integration with respect to time:

$$\begin{aligned} T &= U + H \\ T_0 &= U_0 + H, \end{aligned} \tag{7.15}$$

with the index 0 denoting the initial conditions (or data) and H a constant of integration independent of time and constant for any particular motion, that is for assigned initial data. Hamilton underlined that the Eq. (7.15) is one of the seven "known integrals already mentioned" that he named the *law of living forces* [176].[23]

The 'constant' quantity H may however receive any arbitrary increment whatever, when the system is subject to different initial data. In such a case, by indicating with δ the variation with respect to the initial data, the relation can be written:

$$\delta T = \delta U + \delta H, \tag{7.16}$$

which integrated with respect to time and by using the Eqs. (7.10) and (7.13) give [176][24]:

$$\begin{aligned} &\int \sum m(dx \cdot \delta x' + dy \cdot \delta y' + dz \cdot \delta z') \\ &\qquad = \int \sum m(dx' \cdot \delta x + dy' \cdot \delta y + dz' \cdot \delta z) + \int \delta H \cdot dt. \end{aligned} \tag{7.17}$$

[22] p. 250.

[23] p. 250.

[24] p. 251. Note that Hamilton indicated the derivative with respect to time with the apex$'$.

This can be easily verified by imposing the correspondence term by term. Notice that in the last integral δH is independent of time and can be removed from the integral.

Hamilton at this point introduced a new function V, named the *characteristic function*, defined by the relation:

$$V = \int \sum m(x' \cdot \delta x + y' \cdot \delta y + z' \cdot \delta z) = \int_0^t 2T dt, \tag{7.18}$$

namely the accumulated living force, called often the *action* of the system, from its initial to its final positions.

Hamilton found that:

$$\begin{aligned} \delta V = &\sum m(x' \cdot \delta x + y' \cdot \delta y + z' \cdot \delta z) \\ &- \sum m(a' \cdot \delta a + b' \cdot \delta b + c' \cdot \delta c) + t\,\delta H, \end{aligned} \tag{7.19}$$

where a, b and a', b' or better by making explicit the indices, $a_1, b_1, c_1, \ldots, a_n, b_n, c_n$ and $a'_1, b'_1, c'_1, \ldots a'_n, b'_n, c'_n$, representing the initial values respectively of $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ and $x'_1, y'_1, z'_1, \ldots, x'_n, y'_n, z'_n$,

Hamilton maintained that V could be considered as a function of its final $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ and initial $a_1, b_1, c_1, \ldots, a_n, b_n, c_n$ coordinates and of H. He was quite concise on the point, but later explained how this could be done. Indeed the final positions and velocities can be given as functions of the initial position and velocities:

$$\begin{aligned} x_i &= x_i(a_1, b_1, c_1, \ldots, a_n, b_n, c_n; a'_1, b'_1, c'_1, \ldots a'_n, b'_n, c'_n, t), \quad i = 1, 2, \ldots, n \\ x'_i &= x'_i(a_1, b_1, c_1, \ldots, a_n, b_n, c_n; a'_1, b'_1, c'_1, \ldots a'_n, b'_n, c'_n, t), \quad i = 1, 2, \ldots, n, \\ H &= \frac{1}{2} \sum m x_i'^2 + U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) \\ &= \frac{1}{2} \sum m a_i'^2 + U(a_1, b_1, c_1, \ldots, a_n, b_n, c_n). \end{aligned} \tag{7.20}$$

In this system, at least in principle, x'_i and a_i can exchange their roles; that is x_i and a_i can be considered as independent parameters with which to express x'_i and a'_i, and H.

By differentiation of V, the following expressions are obtained, as clear from Eq. (7.19):

First the, group,

$$\begin{aligned}
\frac{\delta V}{\delta x_1} &= m_1 x_1'; & \frac{\delta V}{\delta x_2} &= m_2 x_2'; & \cdots & \frac{\delta V}{\delta x_n} = m_n x_n' \\
\frac{\delta V}{\delta y_1} &= m_1 y_1'; & \frac{\delta V}{\delta y_2} &= m_2 y_2'; & \cdots & \frac{\delta V}{\delta y_n} = m_n y_n' \\
\frac{\delta V}{\delta z_1} &= m_1 z_1'; & \frac{\delta V}{\delta z_2} &= m_2 z_2'; & \cdots & \frac{\delta V}{\delta z_n} = m_n z_n'.
\end{aligned} \tag{7.21}$$

Secondly the group:

$$\begin{aligned}
\frac{\delta V}{\delta a_1} &= -m_1 a_1'; & \frac{\delta V}{\delta a_2} &= -m_2 a_2'; & \cdots & \frac{\delta V}{\delta a_n} = -m_n a_n' \\
\frac{\delta V}{\delta b_1} &= -m_1 y_1'; & \frac{\delta V}{\delta b_2} &= -m_2 b_2'; & \cdots & \frac{\delta V}{\delta b_n} = -m_n b_n' \\
\frac{\delta V}{\delta c_1} &= -m_1 c_1'; & \frac{\delta V}{\delta c_2} &= -m_2 c_2'; & \cdots & \frac{\delta V}{\delta c_n} = -m_n c_n'.
\end{aligned} \tag{7.22}$$

and finally, the equation:

$$\frac{\delta V}{\delta H} = t. \tag{7.23}$$

So that if this function V were known, it would only remain to eliminate H between the $3n+1$ Eqs. (7.21) and (7.23), in order to obtain all the $3n$ intermediate integrals, or between (7.22) and (7.23) to obtain all the 3n final integrals of the differential equations of motion; that is, ultimately, to obtain the $3n$ sought relations between the 3n varying coordinates and the time, involving also the masses and the 6n initial data above mentioned; the discovery of which relations would be (as we have said) the general solution of the general problem of dynamics. We have, therefore, at least reduced that general problem to the search and differentiation of a single function V, which we shall call on this account the CHARACTERISTIC FUNCTION of motion of a system; and the Eq. (7.19), expressing the fundamental law of its variation, we shall call the *equation of the characteristic function*, or the LAW OF VARYING ACTION [176].[25]

In the following quotation Hamilton underlined that although Lagrange and other scientists possessed the concept of action, they considered its variation by assuming

[25] pp. 251–252. To grasp the meaning of Hamilton's approach I am presenting a simple problem, that of a mass point m falling under the action of a unitary gravity force g. Assuming an axis x directed upward and denoting respectively with x, x' the position and velocity of m at time t, a and a' the position and velocity at time 0, the following relations can be written:

$$H = \frac{1}{2} m x'^2 + mgx = \frac{1}{2} m a'^2 + mga, \tag{7.24}$$

with the characteristic function V given by:

$$V = \int_0^t x'^2 dt = \int_a^x x' dx \tag{7.25}$$

by replacing here the value of x' obtained from the first of the Eq. (7.24) and integrating, the following expression is obtained:

as fixed the initial and final configurations, so that they only obtained the differential equations of motion, while he was obtaining the motion itself.

> Yet from not having formed the conception of the action as a function of this kind, the consequences that have been here deduced from the formula (7.19) for the variation of that definite integral appear to have escaped the notice of Lagrange, and of the other illustrious analysts who have written on theoretical mechanics; although they were in possession of a formula for the variation of this integral not greatly differing from ours. [...] they appear to have deduced from this result only the well known law of *least action*
>
> [...]
>
> But when this well known law of least, or as it might be better called, of stationary action, is applied to the determination of the actual motion of the system, it serves only to form, by the rules of the calculus of variations, the differential equations of motion of the second order, which can always be otherwise found. It seems, therefore, to be with reason that LAGRANGE, LAPLACE, and POISSON have spoken lightly of the utility of this principle in the present state of dynamics. A different estimate, perhaps, will be formed of that other principle which has been introduced in the present paper, under the name of the law of varying action, in which we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system, and (in general) the quantity H, and which serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals [176].[26]

The final equation of the living forces (7.15-1) when combined with the system (7.21) took the new form [175][27]:

$$\frac{1}{2}\sum\frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}=U+H \tag{7.28}$$

(Footnote 25 continued)

$$V(x,a)=-\frac{m}{3}\left\{\sqrt{\left[\frac{2(H-mgx)}{m}\right]^3}-\sqrt{\left[\frac{2(H-mga)}{m}\right]^3}\right\}. \tag{7.26}$$

Notice that notwithstanding the simplicity of the dynamical problem, the expression of the characteristic function is quite complex.

To complete the example also the intermediate and final integrals of motion are found:

$$\begin{aligned}\frac{\delta V}{\delta x}&=m\sqrt{\frac{2(H-mgx)}{m}}=mx'\\ \frac{\delta V}{\delta a}&=-m\sqrt{\frac{2(H-mga)}{m}}=-ma'\\ \frac{\delta V}{\delta H}&=\frac{m}{g}\left[\sqrt{\frac{2(H-mgx)}{m}}-\sqrt{\frac{2(H-mga)}{m}}\right]=t.\end{aligned} \tag{7.27}$$

By solving the last relation with respect to H and substituting it in the other two, the equations of the motion are obtained. In particular the second relation gives x as a function of the initial coordinate a, velocity, a' and t.

[26] p. 252.

[27] p. 253.

and the initial equation of the living forces (7.15-2) when combined with the system (7.22):

$$\frac{1}{2}\sum\frac{1}{m}\left\{\left(\frac{\delta V}{\delta a}\right)^2+\left(\frac{\delta V}{\delta b}\right)^2+\left(\frac{\delta V}{\delta c}\right)^2\right\}=U_0+H. \tag{7.29}$$

For Hamilton these two partial differential equations, initial and final, of the first order and the second degree, must both be identically satisfied by the characteristic function V. They furnish the principal means of discovering the form of that function, and are of essential importance in his theory [176].[28]

> In the method of the present essay, this problem is reduced to the search and differentiation of a single function, which satisfies two partial differential equations of the first order and of the second degree: and every other dynamical problem, respecting the motions of any system, however numerous, of attracting or repelling points, (even if we suppose those points restricted by any conditions of connexion consistent with the law of living force,) is reduced, in like manner, to the study of one central function, of which the form marks out and characterizes the properties of the moving system, and is to be determined by a pair of partial differential equations of the first order, combined with some simple considerations. The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another: and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation [176].[29]

At the end of his long first paper Hamilton found again Lagrange's equation; he also found the other six primitive integrals to add to the living force principle (namely the three laws of the motion of the center of gravity and of areas) and the motion of two or three isolated bodies. After some considerations on approximate solutions and other questions, Hamilton introduced one more fundamental function S, a transformation of the characteristic equation, defined by:

$$V=tH+S, \tag{7.30}$$

or equivalently by the definite integral [176]:[30]

$$S=\int_0^t (T+U)dt \tag{7.31}$$

as a function of the initial and final coordinates and of time. Notice that the integrand of the previous relation is what now is commonly called Lagrangian and usually written as $L=T-\underline{U}$, with $\underline{U}=-U$ the potential energy.

[28] p. 253.

[29] p. 248.

[30] p. 307.

> It is worth observing, that when S is expressed by this definite integral, the conditions for its variation vanishing (if the final and initial coordinates and the time be given) are precisely the differential equations of motion (3),[31] under the forms assigned by Lagrange. The variation of this definite integral S has therefore the double property, of giving the differential equations of motion for any transformed coordinates when the extreme positions are regarded as fixed, and of giving the integrals of those differential equations when the extreme positions are treated as varying [177][32]:

It is clear from its final position in the paper that Hamilton only came to S in the last moment. Indeed in the subsequent second memoir of 1835, S became the chief function of his work and was named the *principal function* of the motion. Hamilton started considering a change of coordinates from $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ to $\eta_1, \eta_2, \eta_3, \ldots, \eta_{3n}$. In this way the kinetic energy in general depends on both η_i and η'_i. Hamilton, following an approach that is now a standard of theoretical mechanics, introduced the *generalized moments* [modern term] [177]:[33]

$$\frac{\delta T}{\delta \eta'_1} = \omega_1; \ldots, \frac{\delta T}{\delta \eta'_{3n}} = \omega_{3n}, \tag{7.32}$$

These relations can be seen as a system of $3n$ equations in $\eta_i, \eta'_i, \omega_i$ which can be solved with respect to η'_i so that the kinetic energy can be given the following form:

$$T = F(\omega_1, \omega_2, \ldots, \omega_{3n}, \eta_1, \eta_2, \ldots, \omega_{3n}). \tag{7.33}$$

Then Hamilton introduced, for ‘abridgement’, the following expression H, today known as the *Hamiltonian function*[34]:

$$\begin{aligned} H = F - U = F(\omega_1, \omega_2, \ldots, \omega_{3n}, \eta_1, \eta_2, \ldots, \omega_{3n}) \\ - U(\eta_1, \eta_2, \ldots, \eta_{3n}). \end{aligned} \tag{7.34}$$

and argued and proved, that the following relations hold true [177]:[35]

---

31 The classical Lagrange equations. The fact that the variation of functions S equated to zero by maintaining fixed the initial conditions gives rise Lagrange equations is commonly referred to as the *Hamilton theorem* [352].

32 p. 99.

33 p. 97. Hamilton’s procedure although correct is today carried out with the use of the Legendre transformation concept [554, p. 167].

34 In the second memoir Hamilton had not yet introduced the symbol H used in the first memoir in which it is not taken for granted it is a constant in time and equal to the total mechanical energy; which was proved subsequently.

35 p. 98.

$$
\begin{aligned}
&\frac{d\eta_1}{dt} = \frac{\delta H}{\delta \omega_1}; \quad \frac{d\omega_1}{dt} = -\frac{\delta H}{\delta \eta_1};\\
&\frac{d\eta_2}{dt} = \frac{\delta H}{\delta \omega_2}; \quad \frac{d\omega_2}{dt} = -\frac{\delta H}{\delta \eta_2};\\
&\dots\\
&\frac{d\eta_{3n}}{dt} = \frac{\delta H}{\delta \omega_{3n}}; \quad \frac{d\omega_{3n}}{dt} = -\frac{\delta H}{\delta \eta_{3n}}.
\end{aligned} \tag{7.35}
$$

These equations, which Hamilton introduced without emphasis, are now considered one of his more important results and are often called the *Hamilton equations of motion* or the *canonical equations* of motion.[36] If the Hamiltonian H is thought of as a function of η_i and η_i', then the equations obtained are those of Lagrange.[37]

> This outstanding system of equations appears for the first time in one of Lagrange's papers (1809) which deals with the perturbation theory of mechanical systems. Lagrange didd not recognize the basic connection of these equations with the equations of motion. It was Cauchy who (in an unpublished memoir of 1831) first gave these equations their true significance. Hamilton made the same equations the foundation of his admirable mechanical investigations. The reference to Hamilton canonical equations is thus fully justified, although Hamilton's paper appeared in 1835 [554].[38]

Beyond statement of his equations, Hamilton was interested in finding their solution and suggested the use of the *principal function S* introduced in the memoir of 1834. It was not difficult for Hamilton to prove the following result [177]:[39]

$$
\begin{aligned}
&\omega_1 = \frac{\delta S}{\delta \eta_1}; \quad p_1 = -\frac{\delta S}{\delta e_1};\\
&\omega_2 = \frac{\delta S}{\delta \eta_2}; \quad p_2 = -\frac{\delta S}{\delta e_2};\\
&\dots\\
&\omega_{3n} = \frac{\delta S}{\delta \eta_{3n}}; \quad p_{3n} = -\frac{\delta S}{\delta e_{3n}}.
\end{aligned} \tag{7.36}
$$

Here e_i and p_i are respectively the values for $t = 0$ of η_i and ω_i.

The principal function S must satisfy the two following equations between its partial differential coefficients of the first order, which offer the chief means of discovering its form [177]:[40]

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[36] Lagrange had already introduced these equations to discuss the perturbation theory in the *Mécanique analytique*, see [212, p. 357].

[37] p. 169.

[38] p. 166.

[39] p. 99.

[40] p. 100.

$$\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \ldots, \frac{\delta S}{\delta \eta_{3n}}, \eta_1, \eta_2, \ldots, \eta_{3n}\right) = U(\eta_1, \eta_2, \ldots, \eta_{3n})$$
$$\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \ldots, \frac{\delta S}{\delta e_{3n}}, e_1, e_2, \ldots, e_{3n}\right) = U(e_1, e_2, \ldots, e_{3n}). \quad (7.37)$$

The difficulty of mathematical dynamics is therefore reduced to the search and study of this one function S, which may for that reason be called the PRINCIPAL FUNCTION OF A SYSTEM [177].[41]

Hamilton's results were criticized by Carl Gustav Jacob Jacobi (1804–1851) whose criticisms are summarized in the *Über die reduction der Integration der partialen Differentialgleichung erster Ordnung Zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichung*of 1838 and the *Volersungen über Dynamik* of 1842–1843 [190]. Jacobi leveled two criticisms at Hamilton's work. He said:

It appears to me that Hamilton has presented his beautiful discovery in a false light, which both complicates and limits its usefulness unnecessarily. His theorem as he has stated it, has al so the disadvantage of being obscure when one does not have his proof in front of one, since one cannot define one function by two partial differential equations without first showing that such a function really exists. By the choice he has made of the special function S the arbitrary constants become the initial values of the coordinates and of the components of the velocities with respect to the coordinate axes; but this is not an advantage since the introduction of these constants ordinarily makes the integral equations more complicated, and since one can transform the integral equations to this form from any other forms. It is perhaps be cause he has always to consider at the same time two partial differential equations that Hamilton has not applied to his theorem the general rules that Lagrange gives in his lectures on the calculus of functions for integrating a non-linear partial differential equation of the first order in three variables; and for this reason, as I shall show in another memoir, results of the greatest interest for mechanics have escaped him. Finally the requirement that the function S after having to satisfy the first partial differential equation satisfies also a second one leads to a restriction in that it excludes the case where the force-function U contains the time explicitly: for this case, in fact, the second partial differential equation is not valid [190].[42] (A.7.5)

Jacobi's criticisms are then, that Hamilton did not know that his equations necessarily have a solution and that the second of the Eq. (7.37) is unnecessary. For Jacobi it was enough to show that a function S exists which satisfies the first equation.

Jacobi also considered the case of non-conservative forces, obtaining a result which is nowadays classical.

Also in the case where there are no force functions, the form is permitted:

$$\frac{dq_i}{dt} = \frac{\partial T}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial T}{\partial q_i} + Q_i, \quad (7.38)$$

[41] p. 99. The former of the Eq. (7.37) is today known as the Hamilton-Jacobi partial differential equation.

[42] vol. 4, pp. 71–72. Translation in [496].

where:

$$Q_i = \sum_k \left(X_k \frac{\partial x_k}{\partial q_i} + Y_k \frac{\partial y_k}{\partial q_i} + Z_k \frac{\partial x_k}{\partial q_i} \right). \tag{7.39}$$

[190].[43] (A.7.6)

7.4 Opening of New Perspectives

Roots for the renovation of mechanics can be largely found in applied mechanics, which in the period mainly concerned motions transmitted by machines. The emergence of applied mechanics was due to the creation of the French polytechnics and to the development of applied research in Great Britain. However because English engineers, such for example the like of John Smeaton (1724–1792), were scarcely interested in theoretical aspects, applied mechanics as a theoretical discipline was born in France, where engineers received an important theoretical training from the new born schools such as the *École des pontes et chaussées* (1747), the *École royale du génie* of Mézières (1748), the *École de mines* of Paris (1783) and mainly the *École polytechnique* (1794). Thanks to the Écoles, France became the leading nation in physics, mathematics and engineering in the first part of the XIX century.

Applied mechanics was, meantime, applied science and engineering. Though a distinction between science, applied science and engineering is largely debatable [353, 317] I assume that it exists and try to characterize applied mechanics as a historical venture. Applied mechanics was born from the need to educate engineers to solve practical problem raised by modern industrial society. Its founders were the likes of Gaspard Monge, Lazare Carnot, Jean Nicolas Pierre Hachette, Gaspard Gistave Coriolis, Claude Louis Navier and Jean Victor Poncelet who only with difficulty could be classified with a unique label as scientists or engineers. The theoretical mechanics which is part of applied mechanics was usually introduced by these scholars in their treatises by using a less developed formal apparatus than that adopted by mathematicians after Lagrange's *Mécanique analytique*. But, mainly, the aim was no longer to validate principles and develop a general theory but to analyze in a rational way practical problems such as improvement of behavior in machines, building of long bridges for railways and large buildings for industrial purposes. And the study of the resistance of a cantilever by no means required less effort or was more simple, for example, than the study of the universe.

One important aspect of applied mechanics was its focus on experiments, a need that increased throughout the whole century and gave rise to the creation of large laboratories close the polytechnic schools. The function of experiments in applied mechanics may be quite different than that in theoretical mechanics. For the latter its main scope was to validate theories; for the former instead the main purpose was that

[43] vol. 8, p. 141. My translation.

of replacing theory when it was scarcely developed. For example in the technological problem of the construction of a bridge for which there was no satisfactory theoretical knowledge, an applied mechanician, that is an engineer, might build a small model and make experiments on it. Or he could test some particular component of the bridge. Only later might the experiences assume a systematic character and make the objective to develop a general theory for bridges.

The engineer of the École polytechnique pursued solutions of practical problems by means of a scientific approach. His difference from the traditional engineer or architect consisted in the way he faced the problems he had to solve; instead of reproducing previous solutions collected in precious handbooks, he looked for a new solution with the aid of some basic concepts of general characters, paid attention for instance to the law of mechanics, to the strength of material, to the costs, etc. Anyway the basic matter in the scientific engineer's curriculum was mathematics, which was essentially analysis (*Calculus*) in the first half of 1800 to become also geometry (projective geometry) in the second half.

The object of applied mechanics was largely the study of the working of machines, among which were those moved by water and wind, and in the second half of the century, by steam; arguments that today are carried on within the framework of *industrial engineering*. There was also another branch however, which looked at structures of civil and military constructions, today framed into *civil engineering*. The industrial engineering course rated Gaspard Gustave Coriolis, Victor Poncelet, Alexis Thérèse Petit, Jean Marie Duhamel, Jean Hachette, Charles Augustin Coulomb; the civil engineering, Navier, Cauchy, Poisson, Gabriel Lamé, Émile Clapeyron, Adhémar Barré Saint Venant and again Charles Augustin Coulomb. The very term mechanics underwent a semantic shift, in some way a return to the past. The prevailing meaning of the term was no longer that of a branch of physics or mathematics, but of a field of technology that concerned modern machines (and structures).

7.4.1 Mechanics of Machines

The study of machines was a very complex subject, too complex to be treated with the idealized mechanics developed by Newton and Euler, and also by Lagrange's analytical mechanics, because of the complexity of the geometry and the need to take into account frictions and deformations of bodies, and very important the breaking of some parts.

Lazare Carnot, though he could observe the behavior of actual machines, developed a theory of machines operating in quite idealized conditions; the first scholar to afford seriously the real behavior of machines and structures, taking into account frictions and shear forces, was Charles Augustin Coulomb (1736–1806) a physicist and engineer scarcely interested in a sophisticated *Calculus* or general theories. In 1773 he wrote a fundamental paper of structural engineering, the *Essai sur une application des règles de maximis et minimis* [99] where there is almost an embarrassment of riches, for Coulomb proceeded to discuss the theory of comprehensive rupture

of masonry piers, the design of vaulted arches, and the theory of earth pressure, for which he developed a generalised sliding wedge theory of soil mechanics that remains in use today in basic engineering practice. This is the paper where Coulomb used *Calculus* to a greater extent than ever to solve some variational problems, showing he mastered mathematics and that its moderate use in later works derived probably from his greater interest in the physical aspects of the various problems. Coulomb gave a correct theory of bending, in which he did not avoid the simpler aspects, in particular those that had never been modeled by previous scholars. As for instance the distribution of shear stresses on a section of a cantilever loaded by transversal forces. The problem of arc breaking, received a new view, still actual. In all these subjects Coulomb took into account friction.

And it is to friction that in 1781 he dedicated a paper, the *Théorie des machines simples* [100], with which he won a prize of the Académie des sciences de Paris, with Lazare Carnot who had only a mention. In the paper he referred to results of his experiments on different bodies sliding on one another, dry or coated with greasy substances. In 1788 he wrote a paper on the efficiency of the work of animals, *Mémoire sur la force de l'homme* [100]. One of the main problems was to find what weight a man or an animal can carry for each unit of travel so that at the end of a day he has raised the highest weight at a given level. Based on simple experiences he was able to write down equations which allowed him to find the maximum. In 1781 Coulomb published also a report on the efficiency of windmills in Holland, *Observations théoriques et expérimentales sur l'effet des Moulins à vent* [100]. For each windmill he derived the work produced, that consumed, at different wind conditions. He concluded his research not with equations or analytical elaboration but with simple considerations:

> We end these considerations with a reflection. What we believe it would be desirable, for the perfection of Mechanics and Arts, to join in a handbook, is a description with figures of the best machines built in Europe. One will add to this description some experiences in the field, like those I have reported for the windmills, but with a greater number and more detailed; one will compare by means of these experience, the effect that each machine produces, with the quantity of action that it consumes, what is the relation to evaluate the degree of perfection. One would have, in this way, an exact measurement to appreciate with facts all the inventions that the authors, without the least knowledge of Mechanics, charge the Academies and the administration, to obtain the privilege to ruin some particular [100].[44] (A.7.7)

In other papers Coulomb studied problems of elasticity (the torsion of a beam by considering nonlinear and viscous effects, 1784), of electricity (the evaluation of the force between two charges, by obtaining what is today called the law of Coulomb, similar to the gravitational law, 1783), of magnetism (by obtaining a law analogous to that valid for electricity, 1789) [430].

A fundamental role in the development of theoretical applied mechanics can be found in Gaspard Monge (1746–1818). Monge taught before at the École militaire du Génie of Mézières then at the École polytechnique; here he contrasted Laplace who advocated a theoretical teaching based on mathematics, asserting that theory and

[44] p. 317. My translation.

practice should have the same dignity and should be taught with a simple language; Laplace's view eventually prevailed and French science became less practical (while English science became more practical).

Monge's more or less direct students included Lazare Carnot, Fourier, Poisson, Prony, Biot, Dupin. He was the first teacher to offer a course on applied mechanics based both on geometry and mechanics. The theory of his course was reported in the fourth part of his *Traité elementaire de statique* published in 1786 after he had left Mézières [249]. Here Monge defined a machine as an instrument to redirect force in the wanted direction by using fixed supports to destroy some unwanted components. It is a definition that moves away from the classical concept of machines, in which they were seen as instrument that could increase power.

> One calls machine any instrument to convey the action of a given force, to a point which is not on its direction, so that the force can move a body which is not directly applied and move it in a different direction of his own.
>
> 122. One cannot usually change the direction of a force but by decomposing this force into two others, one of which is directed towards a fixed point which destroys it with its resistance, and the other acts according to the new direction: this force with alone can produce some effect, is always a component of the first and, depending on the circumstances, it may be either smaller or larger than it. By changing in this way directions and magnitudes of the forces, one can, with the help of a machine, and of points of support to balance two unequal forces that are not directly opposite [249].[45] (A.7.8)

Monge's language about destroyed forces recalls D'Alembert's lost motions. In 1794 Monge conducted his *Cours révolutionnaire* at the École politechnique. In it there were lessons on machines moved by men, animals, flowing or falling water, wind and steam.

Another key figure of applied mechanics was Jean Nicolas Pierre Hachette (1769–1834), who, a former student of Monge, became professor of descriptive geometry at the École polytechnique. Meantime he delivered a separate course on elements of machines, to continue Monge's course. In 1811 Hachette published the *Trait'e elementaire de machines* [507]. In the introduction Hachette payed homage to Lazare Carnot's contribution to the theory of machines.

> I cannot omit to speak in this notice of a book published in 1803 by M. Carnot entitled *Principes fondamentaux de l'équilibre et du mouvement*, 1 vol. 8^0. The last chapter which summarized in a few pages the entire theory of machines and moving forces which are applied to them, is the work of the most profound savant and the most skilled engineer [507].[46] (A.7.9)

In his treatise Hachette mentioned new machines such as the pyréolophore (an internal combustible engine invented by Joseph Nicéphore Niepce (1765–1833) precursor of the daguerreotype) and a fire engine invented by Charles Cagniard de Latour (1777–1859) [507].[47]

[45] pp. 118–119. My translation.

[46] p. XIX. My translation.

[47] p. 144, 149.

The route opened by Monge and Hachette was continued by Jean Victor Poncelet (1788–1867). Poncelet's scientific work was concentrated into two very different areas, corresponding to two successive stages in his career: projective geometry and applied mechanics. In geometry, his work, conceived for the most part between 1813 and 1824, was published between 1817 and 1832. The bulk of Poncelet's work in applied mechanics and technology was conceived between 1825 and 1840. Though Poncelet's geometric studies are fundamental, I here will not give any mention of them, because they are outside of the scope of the present book. With regard to technological innovation, Poncelet's principal contributions concerned hydraulic engines (such as Poncelet's waterwheel), regulators and dynamometers, and various improvements in the techniques of fortification (a new type of drawbridge, resistance of vaults, stability of revetments). Instead of vast syntheses he preferred precise and limited studies, informed by a profound knowledge of the technical imperatives involved. Consequently his original work is to be found in a much greater degree in the realms of organization and improvements than in brilliant innovations. The influence of his thinking—a mixture of the theoretical and the concrete—on the creation of the field of applied mechanics is indicated by the success of his treatises [431].

Poncelet's course on mechanics applied to machines was lithographed for the students at Metz where he in 1825 became professor. The first authentic edition appeared after 1870 as the *Cours de mécanique appliquée aux machines* [302]. In 1829 he published the fundamental text *Course de mécanique industrielle* that after the second edition in 1841 changed slightly the title in *Introduction à la mécanique industrielle* and underwent numerous editions [301].

Poncelet's mechanics was based on the concept of work; for him the essence of machines was nothing but their capability to transfer work, with the implicit assumption that it is usually conserved in the process. At the beginning of his studies he alternated the word *work* with *quantity of action*, but stabilized using *work* following Coriolis [301][48] and contributing to the diffusion of this term. In the following Table 7.1 a list of different locutions used to indicate work referred to by Poncelet is reported [301].[49] Poncelet brought back the conservation of work to the Lagrangian principle of virtual work, "applied to the change of state of bodies" [301].[50] In the *Course de mécanique industrielle* Poncelet set out a quite generalized form of the principle of living force/work. The principle was not new in itself, it could for instance be found in Lazare Carnot's or even in Lagrange's textbooks. What was new was the relevance that Poncelet gave to it, arriving at a definition of mechanics as the science of the work of forces:

> Loss or gain of living force suffered, between two whatever instants, by a body whose motion varies, is double the amount of work developed in this interval, by the inertia of the body or the driving force directly opposite and equal [301].[51] (A.7.10)

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[48] p. X

[49] p. 65.

[50] p. X.

[51] p. 128. My translation. Poncelet while adopting the nomenclature work from Coriolis, still considered the living force as mv^2 and not $1/2mv^2$.

Table 7.1 Different locutions used to indicate work

Smeaton	Mechanical power
Lazare Carnot	Moment of activity
Monge and Hachette	Dynamic effect
Coulomb, Navier	Quantity of action

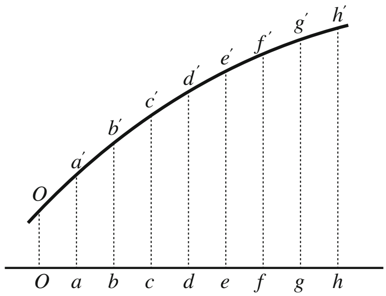

Fig. 7.1 Experimental evaluation of the work of a not constant force (Redrawn from [301, Plate I, Fig. 23])

Poncelet pointed out that the living force could be converted again into work, and living force could be considered as a form of stored work. So Bernoullis' reasoning that saw the potential energy as stored living force was completely reversed. This way to consider work was clearly expressed by Henry Moseley also, in 1841, when he pretended to give a new meaning to the expression *living forces* as the work accumulated in the moving part of a body/machine equal to the difference between the work of the forces that tend to accelerate the motion and the work of the forces that tend to retard the motion [250].[52]

Poncelet spent many pages in the definition of work and suggested also the way to calculate it in experimental situations. For instance he indicated that the work of active and passive forces could be measured from diagrams of force-displacement also in the case of force varrying along the path. The work is measured by the area subtended by two perpendiculars representing the distances along which the force acted and the force itself. The area could be calculated numerically by means of a series of trapezoids as indicated in Fig. 7.1.

> Draw on a plane or table a curve $O'a'b'c'\dots$ the abscissas of which Oa, Ob, Oc, ... represent paths successively described by the point of action of the resistance, and the ordinates represent, according to a suitable scale, the corresponding resistance or the corresponding efforts measured in kilograms. Suppose $Oa, ab, bc, \dots$ be equal and very small spaces described at any time. The partial work having as a measure the products of these small spaces by the corresponding average resistance, assumed as constant on each of them, that is to say, the products $\frac{1}{2}(OO' + aa') \times Oa$, $\frac{1}{2}(aa' + bb') \times ab$, $\frac{1}{2}(bb' + cc') \times bc, \dots$, this work will be represented by the areas of the trapezoids $OO'a'a$, $aa'b'b$, $bb'c'c, \dots$, and the total work will given by the surface of these small trapezoids together [301].[53] (A.7.11)

[52] p. 287.

[53] p. 58. My translation.

Table 7.2 Poncelet's *Introduction a la mécanique industrielle* 1841

Fundamental principles
1. General notions of the physical properties of bodies
2. Preliminary notions on motion and forces
3. Motion due to constant forces
4. Change of work into living force
5. Inertia and living force
6. Communication of motions by impact
7. Motion due to the expansion of gases
8. Mechanical work produced by steam
9. Mechanical work developed by animals
Resistance of solides
10. Preliminary notions on the structure of bodies
11. Notions and principles on the resistance of prisms to traction or compression
12. Experiences on the resistance of solids
13. Examen of the vibrations of prisms under constant loads and impact
14. Consequence and applications concerning the motion of prisms
Frictions of solids

Table of content [301]

There is no doubts that Poncelet, more clearly than Carnot and the Bernoullis, had an intuitive grasp of the conservation of energy, and not only in its mechanical aspects:

> However, the heat that is counted among the mechanical forces, and the electricity which is also a force which develops, such as the heat, by percussion either by friction or even by simple contact of different bodies [301].[54] (A.7.12)

Table 7.2 shows the content of the *Introduction a la mécanique industrielle* (limited to solid bodies).

7.4.1.1 Textbooks on Applied Mechanics

The courses in mechanics of machines were preceded by courses of general mechanics. They were delivered at the École polytechnique, generally by professors who also taught mathematical analysis. The following Table 7.3 shows a list of professor of mechanics/mathematics in the years 1794–1855 [501].[55]

The approach in teaching was not the same for all professors, though all basically followed Lagrange's analytical mechanics. Attention, however, was also payed to themes ignored by Lagrange, such as those of impact and friction.

[54] p. 252. My translation.

[55] p. 235.

Table 7.3 *Professeurs* of analysis and mechanics at the École polytechnique

1794–1815	de Prony
1816–1828	Ampère
1828–1839	Mathieu
1839–1850	Liouville
1851–1868	Duhamel
1794–1799	Lagrange
1799–1808	Lacroix
1808–1815	Poisson
1816–1830	Cauchy
1831–1836	Navier
1836–1839	Duhamel
1840–1855	Sturm

Siméon Denis Poisson

Among the professors of mechanics one of the most influential was for sure Siméon Denis Poisson whose *Traité de mécanique* remained long a reference for students of mechanics in France and abroad. The first edition was of 1811 into two volumes of more than 1,000 pages and a run of 5,000 copies [292]. The second edition of 1833 had a substantial increment in pages (1,500) and the same run [296]. A third edition of 1838 has a very compact format in a unique volume with less than 500 pages in total [297].

Though Poisson was a skillful mathematicians and sometimes used sophisticated techniques, his book was thought to meet the needs of students. The departure from Lagrange's *Mécanique analytique* was evident just leafing through the pages for the presence of numerous drawings. However Poisson's main instruments for theoretical analysis were those of Lagrange; the principle of virtual work in statics and the same principle added to D'Alembert's principle in dynamics. Table 7.4 reports the list of the chapters of Poisson's treatise.

The table shows the didactic nature of the treatise. The main concepts are presented in a simple way before giving a general law. Notice that in the chapter devoted to the composition of moments Poisson made no reference to the theory of couples that had recently (1803) been introduced by Louis Poinsot in his *Élement de statique* [290]. There is instead a great space devoted to the dynamics of solids bodies, representing concepts like moment of inertia as independent quantities endowed with a physical meaning, and not simply coefficients of the equation of motion as they were in Lagrange. A major departure from Lagrange's mechanics is evident from the stressing on collision of bodies, the use of an engineering approach to hydraulics and the study of elastic bodies.

Poisson's treatment of impact was of a certain relevance for the theory of machines. He rejected the model of a perfectly hard body to accept that of a hard-plastic body (even accepted by Poncelet and justified by the molecular conception of matter), that is a body that during the impact suffers an infinitesimal deformation that is not recovered:

Table 7.4 Poisson *Traité de mécanique* 1833

Statics part I
1. The composition and balance of the forces applied to the same point
2. On the balance of the lever
3. On the composition and balance of the parallel forces
4. General considerations on heavy bodies and centers of gravity
5. Centres of gravity
6. Calculation of the attraction of bodies
Dynamics part I
1. The straight motion and the measure of forces
2. Examples of rectilinear motions
3. The curvilinear motion
4. On the centrifugal force
5. Examples of the motion of a point on a curve or on a given surface
6. Example of the motions of a completely free mobile
7. Digression on the universal attraction
Statics part II
1. Of the equilibrium of a solid body
2. Theory of the moments
3. Equilibrium of elastic bodies
4. Principle of virtual velocities
Dynamics part II
1. General principle of dynamics
2. Determination of the moments of inertia and principal axes
3. Motion of a solid body around a fixed axis
4. Motion of a solid body around a fixed point
5. Motion of a free solid body
6. Motion of a solid heavy body on a plane with regard to the friction
7. Impact of bodies with any shape
8. Examples of the motion of a flexible body
9. Equation and general properties of the motions of a system of bodies

Table of content (limited to solid bodies)

How much hard the two mobiles, they are always more or less compressible. Because of the difference in their velocities v and v', they will be compressed, in leaning on each other and, during this compression, the velocity of one of the two bodies, m for example, will decrease by infinitesimal degrees, and that of m' increase until these two velocities have become equal. However, from this moment, there are two separate cases to be considered.

1- If the two spheres are entirely deprived of elasticity them will cease to act on each other when their speeds become equal and will continue to move with a common velocity remaining joined and maintaining the shape that the compression has impressed to them [296].[56] (A.7.13)

[56] p. 27. My translation.

With respect to the analysis of elastic bodies it must be said that Poisson was one of the advocates of the theory of elasticity based on the molecular model and he was the promotor of a new kind of mechanics, which considered bodies no longer idealized as rigid elements but as elements passible of deformation. He referred to the new approach as to the *Méanique physique*.

> Lagrange has gone as far as we can conceive, when he replaced the physical constraints of body with the equations between the coordinates of their points; what constitutes the *Mécanique analytique*, but besides this admirable conception we could now raise the *Mécanique physique*, the single principle of which would be to bring anything to the molecular actions, which transmit from one point to another action of the assigned forces, and are the means of their balance. In this way, there would be no need of special assumptions when one wants to apply the general rules to specific mechanical issues [291].[57] (A.7.14)

Poisson reformulated the principle of conservation of living forces with an important semantic change; it simply became the *principle of living force*, with no stress on conservation that for him did not subsist, and he no longer adopted the terminology, still present in Lazare Carnot, of latent living force.

In a chapter devoted to the central forces (Sect. 9.4, Book 4, vol. 2), Poisson considered the living forces of a set of mass points at two different moments separated by a finite interval of time. The value of the living force at the later moment is different than at the first moment; that is, a variation of living force is gradually taking place because of the action (the work) of the (central) forces. The value at the first moment is mv^2 and at the later one mk^2; the mass of the bodies remains constant and the interval of time has a finite duration. As the system of force was central, the work (modern term) on an infinitesimal displacement is an exact differential, that is can be derived by variation from a function φ of the coordinates only of the points of the system. In such a case the variation of the living force can be expressed as the difference of two values of a function depending on the configuration of the mass points only:

> It will result at a whichever instant, $\sum mv^2 - \sum mk^2 = 2\,\varphi(x, y, z, x', \text{etc.}) - 2\,\varphi_2(a, b, c, a', \text{etc.})$- The quantities mv^2 and mk^2 are the sums of the livings force of all the points of the system at this instant and at the beginning of the motion; this equation implies that the difference of these two sums only depends on the coordinates of the mobile bodies and in no way on their constraints nor the paths which they have followed in order to pass from the initial positions to those that they occupy at the end of a time t. It is in this that one finds the law of motion to which it was given the name of the principle of the living forces.
> […]
> If all the points of the system occupy the same position in two different instants, the sums of their living forces will be the same at these instants [296].[58] (A.7.15)

The important consideration here is that the value of the variation of the living forces between two different configurations of the system of points is independent of the path, a concept previously introduced by Lazare Carnot in a less general situation [629].[59]

[57] p. 361. My translation.

[58] vol. 2, pp. 478–479. My translation.

[59] p. 165.

> Finally, if one uses what machines he likes, even with springs, provided that in this case, to deliver the springs in the same state of tension where they took the first moment, the moment of activity that the external agent, employed to move this system, consume to produce this effect, will be the same, assuming that the system is at rest at the first instant of the movement, and at the last [69].[60] (A.7.16)

Gaspard Gustave Coriolis
The most important theoretical contributions on mechanics applied to machines in the first part of the XIX century are usually individuated in the books by Gaspard Gustave Coriolis (1792–1843), the *De calcul de léffect de machines*, published in 1829 [94] and the *Traité de la mécanique des corps solides*, published in 1844, with this latter that can be considered a reworking of the former [98]. In the more than twenty years which separate Coriolis' treatises from Carnot's *Principe fondamentaux de l"equilibre et du mouvement* no important intermediate work was published and so Coriolis' texts are inevitably connected to Carnot's. Actually Carnot's book was followed by a large number of professional works to deal with the theory and practice of power technology and mechanics published in France, the products perhaps of revolutionary temper and the establishment of the l'École polytechnique, but they added little from a theoretical point of view. Thus, in 1790 Gaspard Clair François Marie Riche de Prony (1755–1839) published his *Nouvelle architecture hydraulique* [303] and this was followed, in 1810, by the *Essai sur la science des machines* of André Guenyveau (1782–1861) [174]; in 1811, by the *Traité elementaire des machines* of Jean Nicolas Pierre Hachette [507]; in 1810 Pierre Simon Girard (1765–1836) translated some Smeaton's memoirs which, to judge by frequent references, made a great impression [321]; in 1819 Navier delivered a much revised and corrected version of the first volume of the *Architecture hydraulique* by Bernard Forest de Belidor (1698–1761) [32]; in 1822 came the *Traité de mécanique industrielle* by Gérard-Joseph Christian (1778–1832) [86]; and shortly after in 1829 the *Course de mècanique industrielle* by Poncelet [300] and the *Géométrie et mècanique des arts et métiers et des beaux arts* by Pierre Charles François Dupin (1784–1873) [108]. All these books were largely influenced by Carnot's approaches: the use of work as preferred physical magnitude and the principle of its conservation as a preferred tool of analysis.

It is certain that Carnot had been a very deep thinker; but mainly he was a 'lucky' man who had a simple idea which turned out to be fundamental for applied mechanics. This notwithstanding, his first book, the *Essai sur les machines en générale* of 1782, did not receive great appreciation by applied and theoretical mechanicians. Lagrange, for instance, in his *Théories de fonctions analytiques* of 1797 cited Carnot's theorem as an important achievement [211][61]; but this was only one of the many citations by Lagrange. Probably the reason of Carnot's influence on the subsequent generation of engineers and scientists and on Coriolis too was due to his military and political standing and his role in the foundation of the École polytechnique together with his

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[60] p. 98. My translation.

[61] pp. 273.

direct contact with French scientists [629],[62] [492]. Anyway Carnot's second book, the *Principe fondamentaux de l'équilibre et du mouvement* of 1803, became to be generally known.

Let return to Coriolis' treatises. They differed from Carnot's in some important points. They were didactic books so most parts were given a much greater extension; though considering machines in general it gave large space to particular kinds of machines; moreover terms and concepts were more clearly stated. In the following I will refer to both the editions (1829 and 1844); the former because of its historical priority and its greater stress on machines, the latter mainly for aspects regarding the nomenclature.

In the first chapter of the treatise of 1829, Coriolis defined the main concepts of mechanics. Among them that of force, mass and work. As for most scientists, for him the ontology of force was no longer a problem and he was not interested in its status but only in its use; moreover he left the problem of impact as a special problem and mainly considered forces of continuous nature:

> In what we are going to say word force will apply only to what is analogous to weight, that is to what is called, in most cases, pressure, tension, and traction. With this meaning force could not make the direction and the value of velocity to change sharply without it passes through all the intermediate states [94].[63] (A.7.17)

The mass of a body was defined as the ratio between force and acceleration and its measure was given by the weight of the body at an assigned level over the sea. The concept of work was considered the most important one for the study of machines in motions, while force maintained the prominence for machines in equilibrium. Coriolis introduced the word *work* to indicate what Carnot called moment of activity.

> Those different expressions so vague are not capable to spread out. We will propose the name of dynamical work, or simply work, to the quantity [...]. This name is then very suitable to indicate the union of these two elements, distance and force [94].[64] (A.7.18)

Coriolis used the word *work*, to indicate work, also in subsequent studies, particularly in the *Mémoire sur la manière d'établir les differéns principes pour des systèmes de mécanique des corps, comme en des assemblages de considérant the molecules* of 1835 [96]. Such use he definitely consolidated with his work *Traité de la mécanique des corps solides* of 1844 where, in the preface, he wrote:

> I employed in this work some new nomenclature: I name work the quantity usually named *puissance mécanique*, *quantité d'action* ou *effet dynamique*, and I propose the name *dynamode* for the unity of measure of this quantity. I introduced also one more little innovation by naming living force the product of the weight by the height associated to the height. This living force is one half of the product that today is associated to this name, that is the mass by the square of speed [98].[65] (A.7.19)

[62] p. 107.

[63] pp. 2–3. My translation.

[64] p. 17. My translation.

[65] p. IX. My translation.

Notice that Coriolis is introducing the factor $1/2$ in front of the expression of the living force (that is kinetic energy), because he suggested to measure the living force of a body of mass m and velocity v with the product mh, with h the height the body can reach if thrown upward with an initial velocity v ($h = v^2/2$).

In a note to the passage above Coriolis wrote:

> This term work is so natural in the sense that I use it, which, though it has never been either proposed or approved as a technical expression, nevertheless it was used accidentally by Mr. Navier in his notes on Belidor and Prony in his *Mémoire sur les expériences de la machine du Gros-Caillou* [98].[66] (A.7.20)

Although Coriolis's texts were fundamental to the spread of the term work, again, at the end of the XIX century propositions like: *principle of virtual velocities, principle of moments and principle of virtual work*, co-existed. See on the purpose a note by Saint-Venant in his translation of Alfred Clebsch's text on the theory of elasticity, where one speaks of a theorem of virtual work or virtual velocities [91, 92].[67] Notice that Coriolis himself used the expression principle of virtual velocities to indicate the principle of virtual work. Principle which was at the basis of his mechanics and was associated to D'Alembert's principle to obtain all the necessary equations of motion.

In the first chapter of the text of 1829 Coriolis introduced machines in general, in an interesting way, very close to Carnot's:

> Here in after we will use the name machine to indicate the mobile bodies to which we will apply the equation of living forces: in this sense a single body which moves is a machine, so does a more complicate system. In each particular case, once we will know by what bodies in motion the machine is composed it will be enough to apply the principle previously established, to know precisely what are the masses which must be considered in the living forces evaluation, and what are the motive and resistant forces which must be considered to evaluate the amount of work [94].[68] (A.7.21)

In the second chapter, he paused to analyze the evaluation of the work which can be obtained by different natural agents. Here Coriolis considered also the case of friction and of elasticity and proposed an evaluation of the work consumed by friction in particular cases, for instance that of rotating gears. In the third part of the book Coriolis considered the case when instead of a system of mass points there was a system of extended bodies, for which he assumed a molecular model, accepting Poisson's idea of the mécanique physique.

Here the language is similar to that found in modern treatises of practical engineers [629] and those based on thermodynamics, to appear in a few years. The virtual work assumed a degree of reality. It was more a physical quantity, observable and measurable in some way, than a purely mathematical definition as it appeared in the works of Lagrange and his immediate successors. This approach was also kept in subsequent works:

[66] p. IX. My translation.

[67] p. 577.

[68] p. 20. My translation.

If the equilibrium is obtained under the action of forces P, each molecule will be in equilibrium and, taking into account all the molecular actions, it will be:

$$\sum R\,\delta r + \sum P\,\delta p = 0$$

If now a virtual motion of each body is considered that leaves its invariability or solidity, and yet in this motion the bodies are left to slip or turn over each other with the freedom of motion allowed by the machine constitution, it is found that a large portion of virtual works $R\,\delta r$ vanishes: it is that due to actions between molecules that have not switched away during the virtual motion, namely those belonging to the same body. In the equation above it will remain only the element of virtual work $\sum P\,\delta r$ coming from the actions among the molecules of adjacent bodies, when in the virtual motion these bodies do not move together as one system, but they slip or roll on each other. The actions R that remain will be only due to molecules that are at a distance from the contact surface less than the extension of the molecular actions, or in other words, the radius of the sphere of action [96].[69] (A.7.22)

The role of *work* as a real entity, a substance, is more clearly evident in the following quotation of the 1844 edition of the *Traité de mécaniques des corps solides*:

One can compare the transmission of work for machines with the flow of a fluid [emphasis added] which is spread throughout the bodies, passing from one to the other by the contact points. It would be divided into several streams, where a single body will push more bodies; on the contrary a reunion of several streams will occur when several bodies push one. This fluid could also accumulate in some body and stay in reserve until that new contacts, or contacts with greater flow, made exit in a greater amount: this reserve of work, which we assimilate to a fluid, is what we called the *living force*.[70] Still following this comparison, a machine, in the ordinary sense of the word, is a set of moving objects arranged to form a kind of channel by means of which the work takes its course to pass, as integrally possible to the points where it is needed. It is gradually lost by friction and deformation of the body, or it is spread into the earth, where, extending indefinitely, it soon becomes insensitive [98].[71] (A.7.23)

Coriolis observed that in the impact of elastic bodies the theorem of the living forces is not always valid, because a part of the energy is absorbed as oscillatory energy, which dissipates in the environment [98].[72]

Coriolis examined the problems related to friction, using a language that was completely different from the classic language used for virtual work laws. For instance in his *Mémoire sur la manière d'établir les differéns principes pour des systèmes de mécanique des corps, comme en des assemblages de considérant the molecules* of 1835 he suggested that, when by the nature of bodies, there is no possibility of sliding and one body rolls on another, "the virtual velocities become zero for the points of contact […] so that the sums of the elements of work due to this rolling are zero" [96].[73] Finally he concluded:

We are led to realize that the principle of virtual velocities in the equilibrium of a machine, composed of more bodies, cannot take place without considering the sliding friction, when

[69] pp. 114–115. My translation.

[70] Note the persistence of the expression living force to indicate the potential energy.

[71] p. 117. My translation.

[72] p. 107.

[73] p. 116.

> the virtual displacements cause the sliding of the the bodies one on others, and also that the rolling, when bodies cannot take that virtual motion without deformation near the contact points.
> Frictions are recognized always, for experience, able to maintain equilibrium in a certain degree of inequality between the sum of the positive work and the sum of the negative work, here taking as negative the elements belonging to the smaller sum. It follows that the sum of the elements to which they give rise has precisely the value that can cancel the total sum and is equal to the small difference between the sum of the positive and negative elements [96].[74] (A.7.24)

So friction contributes to the balance of the work by providing a negative term, since "for experience, it gives raise to a negative sum."

The final part of the *Traité de la mécanique des corps solides* (Chap. IV) is quite applicative. Coriolis considered the efficiency of machines such as windmills and water mills by varying the speed of the fluid and the size and shape of the paddles and their speed of rotation. He studied the role of flywheels to improve efficiency by regulation of motion. He also devoted a few interesting pages to the work and efficiency of thermal machines that use steam, without any reference however to Sadi Carnot's book *Réflexions sur la puissance motrice du feu et sur les machines* of 1824 [73].

Coriolis' treatise ends with a series of tables referring to results of the dissipation of work to produce different useful effects deduced from different authors, among which were Navier and Hachette [98].[75] %

7.4.2 Mechanics of Structures

In his *Mécanique industrielle* Poncelet, besides the study of kinematics and dynamics of machines, also addressed the problem of their resistance, thus associating mechanics of machines to mechanics of structures. This second branch of applied mechanics had the aim to study the strength and deformability of structural elements, with respect to both machines, subjected to dynamical actions, and civil constructions, subjected mainly to static actions. For many reasons, of which I will mention only the simplest, the mechanics of structures were received, at the beginning, with great enthusiasm by 'civil engineers'. Also at the beginning mainly linear elastic behavior was considered. This caused the developments of the theory of elasticity, that first received attention from engineers (and physicists) to become a preferred matter for mathematicians, once the principles were made precise.

7.4.2.1 The Classical Molecular Model

The explanation of the elastic behavior of materials at the beginning of the XIX century was based on a corpuscular structure of matter. One of the champions of this view

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[74] p. 117. My translation.

[75] pp. 251–255. The unit of measure is that he unsuccessfully proposed, the *dynamods*.

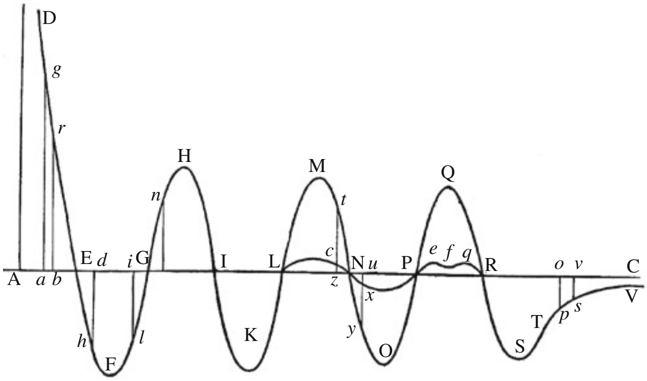

Fig. 7.2 Corpuscular model: force between corpuscles versus their mutual distance (Redrawn from [58, Table I, Fig. 1])

was Pierre Simon Laplace (1749–1827) who considered matter to be shaped as small hard corpuscles endowed with extension and mass [224].[76] Taking inspiration from Newton's *Opticks* of 1704 [267], for Laplace the intermolecular force is repulsive at very short distances and becomes attractive for greater distance, tending in the limit to the law of the inverse square. One interesting variant to Laplace's approach was that due to Ruggiero Giuseppe Boscovich (1711–1787), for whom the corpuscles became mass points, deprived of extension but endowed with mass, to be assumed as centers of forces [57, 58]. Figure 7.2 shows the complex trend of force between two particles (vertical axis) as a function of their mutual distance (horizontal axis) considered by Boscovich.

According to Laplace (and Boscovich) and most French scientists in the early 1800s, every physical phenomenon must and could be explained by the laws of particle mechanics, perfecting Descartes' mechanistic view [466]. The substance of material bodies has a discrete structure and the space is pervaded by thin particles, which make up the *ether*. All physical phenomena propagate in space by a particle of ether to its immediate neighbor through impact or forces of attraction or repulsion. This point of view allows one to overcome the difficulties of the concept of action at a distance: how—the physicists of the time asked—can two bodies interact, for example, attract, without the intervention of a medium? To each physical phenomenon corresponds a state of tension in the ether, propagated by contact.

With the beginning of the XIX century the need was felt to quantitatively characterize the elastic behavior of the bodies and the mathematical theory of elasticity was born. Its introduction was thought to be crucial for an accurate description of the physical world, in particular to better understand the phenomenon of propagation of light waves. The choices of physicists were strongly influenced by mathematics in vogue at that time, namely the differential and integral calculus. It presupposed the mathematics of continuum and therefore found it difficult to marry with the discrete particle model, which has become dominant.

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[76] vol. 4, pp. 349, 350.

Most scientists adopted a compromise approach that today can and is interpreted as a technique of homogenization. The material bodies, with a fine corpuscular structure, are associated with a mathematical continuum C, as may be a solid of Euclidean geometry. The variables of displacement are represented by a regular function **u** defined in C, that assumes meaningful values only for those points P of C that are also positions of corpuscles. The derivatives of the function **u** with respect to the variables of space and time also have meaning only for the points P. The internal forces exchanged between the corpuscles, at the beginning thought to be concentrated, are replaced by their average values that are attributed to all the points of C, thus becoming stresses σ. Other scientists gave up the corpuscular physical model considering it only in the background. They founded their theories directly on the continuum, whose points had now all ‘physical’ meaning. On the continuum are defined both the displacements and the stresses, as had already been done in the XVIII century by Euler and Lagrange for fluids. Some scientists oscillated between the two approaches, among them Augustin Louis Cauchy (1760–1848) (but the Italian Gabrio Piola (1794–1850) was in a similar position [385]) who, while studying the distribution of internal forces of solids, systematized mathematical analysis, dealing with the conception of infinite and infinitesimal. His oscillations in mathematical analysis were felt by his studies on the constitution of matter [421, 422].

The theories of elasticity of the early XIX century were based on various corpuscular assumptions, introduced almost simultaneously by Fresnel, Cauchy and Navier [75, 76, 148, 252]. French scientists adopted the single word *molecule* for the corpuscles, which lived long in European scientific literature, often flanked by *atom*, without the two terms necessarily having a different meaning, at least until when the studies of the chemical constitution of matter advanced and the terms atom and molecule assumed precise technical meanings which differentiate the areas of application.

Fresnel studied the propagation of light in the ether, thought to be formed by molecules which exchange elastic forces. In a work of 1820 he obtained very interesting results. The first systematic work on the equilibrium and the motion of three-dimensional elastic bodies was however due to Navier, who in 1821 read before the Académie des sciences de Paris an important memoir published only in 1827 [252].

Navier, referring explicitly to Lagrange’s *Méchanique analitique* [209], wrote the equations of local balance of forces acting on an elastic body, thought of as an aggregate of particles that attract or repel each other with an elastic force variable linearly with their mutual displacements.

> One considers a solid body as an assemblage of material molecules placed at a very small distance [from each other]. These molecules exert two opposite actions on each other, that is, a proper attractive force, and a repulsive force due to the principle of heat. Between one molecule M and any other M' of the close molecules there is an attraction P which is the difference of these two forces. In the natural state of the body all the actions P are zero or reciprocally destroy, because the molecule M is at rest. When the body changed its shape, the action P took a different value Π and there is equilibrium between all the forces Π and the forces applied to the body, by which the change of the shape of the body was produced [252].[77] (A.7.25)

[77] pp. 375–376. Our translation.

Navier obtained the equations of equilibrium with the use of the principle of virtual work [252].[78] He followed the approach, already mentioned, common to all French scientists of the XIX century, by considering the body as discrete when he wanted to study equilibrium, while as continuous when he came to describe the geometry and kinematics and obtained simple mathematical relationships, replacing the summations with integrals.[79] Note that in the work of Navier the concept of stress, which was crucial to the mechanics of structures developed later, was not present.

In the French academic world, because of the influence of Laplace's teaching the molecular model soon became dominant. In October 1827 Poisson and Cauchy presented, before the Académie des sciences de Paris, two similar memoirs,[80] in which the molecular model of Navier was reconsidered. In two other memoirs read again before the Académie des sciences de Paris in April 1828 [291] and in October 1829 [295], Poisson made clear the main assumptions of the molecular model and introduced the concept of stress considered as an internal force.

However, it was Cauchy who set the problem in substantially the modern way, by perfecting the concepts of stress, strain and finally the link between stress and strain, or constitutive relationship. Regarding stress, Cauchy adopted the definition given by Poisson, in which reference is made to the mutual forces between the molecules through an infinitesimal surface, and moved in the direction of homogenization, passing from the molecular model of matter to a continuum mathematical model where stresses and displacements are defined as continuous functions.

> Let M be a point in the inner part of the body, at a sensible distance from the surface. Let us consider a plane through this point, dividing the body into two parts, which we will suppose horizontal [...]. Let us denote by A′ the upper part and A the lower part, in which we will include the mass points belonging to the plane itself. From the point M considered as a center let us draw a sphere including a very large amount of molecules, yet the radius of which is in any case negligible with respect to the radius of the molecular activity. Let ω be the area of its horizontal section; over this section let us raise a vertical cylinder, the height of which is at least the same as the radius of molecular activity; let us call B this cylinder; the action of the molecules of A′ over those of A, divided by ω, will be the *pressure* [emphasis added] exerted by A′ over A, with respect to the unity of surface and relative to the point M [291].[81]
> (A.7.26)

With reference to Fig. 7.3 consider a cylinder B having an infinitesimal base ω on a plane perpendicular to an assigned versor **n**, located in the half space A. Let $\mathfrak{m}$ be the molecules inside the cylinder and m those located in the half-space A′ in the

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[78] p. 384.

[79] The difficulty of replacing the summations with the integrals has been the subject of many comments of French scholars, especially Poisson and Cauchy.

[80] See [253, p. clv, clix]. Cauchy's memoir appeared first, with the title *Mémoire sur l'équilibre et le mouvement d'un système de points materiels sollecités par forces d'attraction ou de répulsion mutuelle* [79]. Poisson's memoir appeared with the title *Note sur les vibrations des corps sonores* [293].

[81] p. 29. My translation. Stress was indicated by French scientists by *pressure* or*tension*.

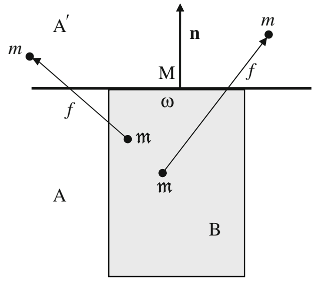

Fig. 7.3 The homogenization of stress according to Poisson

same side of **n**.[82] The force exerted on $\mathfrak{m}$ by all the molecules m is characterized by the three components [80]:[83]

$$\sum \pm \mathfrak{m}m \cos\alpha f(r); \quad \sum \pm \mathfrak{m}m \cos\beta f(r); \quad \sum \pm \mathfrak{m}m \cos\gamma\, f(r), \qquad (7.40)$$

where f is the force between two molecules m and $\mathfrak{m}$ at a distance r, α, β, γ are the director cosines of the radius vector r with respect to an arbitrary coordinate system and the sum is extended to all the molecules m of the half space A′ opposite to the cylinder, or rather to all those in the sphere of molecular action (the sphere defined by the radius of molecular action) of $\mathfrak{m}$. To obtain the force exerted on the cylinder and, according to Poisson, the pressure on the surface ω, the summations should be extended to all the molecules $\mathfrak{m}$ of the cylinder and divided by ω.
Cauchy showed that the vector stress (p) on a generic plane can be determined as soon as one knows the components of the vectors stress (p', p'', p''') on three orthogonal planes, respectively yz, xz, xy. Cauchy introduced the symbols [77]:[84]

$$\begin{cases} A = p' \cos\lambda' \\ B = p' \cos\mu'' \\ C = p' \cos\nu'' \\ D = p'' \cos\nu'' = p''' \cos\mu''' \\ D = p''' \cos\lambda''' = p' \cos\nu' \\ D = p' \cos\mu' = p'' \cos\lambda'' \end{cases} \qquad (7.41)$$

where $\lambda', \mu', \nu', \lambda'', \mu'', \nu'', \lambda''', \mu''', \nu'''$ are the angles of p', p'', p''' with respect to the axis x, y, z respectively. A, B, C are the components of the stresses orthogonal to the coordinate planes, D, E, F are the tangential components which satisfy the reciprocity relations, such for instance $p'' \cos\nu'' = p''' \cos\mu'''$, derived from the equilibrium equations.

In the modern theories of continuum mechanics a fundamental concept is that of constitutive relationship, namely the function which relates the internal forces

[82] The molecules indicated by B, M by Poisson.

[83] p. 257.

[84] p. 68.

(stresses) to the deformations (strains). To find the constitutive relationship Cauchy elaborated the expressions (7.40). Assuming very small displacements, the function $f(r)$ can be linearized to obtain a linear relationship between the stress and the displacement of the molecules. Cauchy showed that this relationship is completely defined by six appropriate combinations of the partial derivatives of the components of displacements ξ, η, ζ of the molecules with respect to their positions x, y, z, in particular:

$$\begin{aligned}&\partial\xi/\partial x,\ \partial\eta/\partial y,\ \partial\zeta/\partial z,\\&(\partial\xi/\partial y+\partial\eta/\partial x),\ (\partial\xi/\partial z+\partial\zeta/\partial x),\ (\partial\eta/\partial z+\partial\zeta/\partial y)\end{aligned} \tag{7.42}$$

which assumed the role of the components of the strain.

In the case of an isotropic behavior of the matter, or equivalently a radial symmetry of the disposition of molecules around each molecule—a situation which can be assumed for many substances, such as steel, stone, etc.—Cauchy arrived to a linear relationship between stress and strain expressed by means a unique constant of proportionality k [81]:[85]

$$\begin{aligned}&A=k\frac{\partial\xi}{\partial x},\quad B=k\frac{\partial\eta}{\partial y},\quad C=k\frac{\partial\zeta}{\partial z},\quad D=\frac{1}{2}k\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right),\\&E=\frac{1}{2}k\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right),\quad F=\frac{1}{2}k\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right).\end{aligned} \tag{7.43}$$

The molecular model by Navier, Cauchy and Poisson—hereinafter referred to to as the classical molecular model—was accepted by the scientific international community, especially in France, because of the simplicity of the theory and the physical basis universally shared. However its conclusions were slowly but inexorably falsified by the experimental evidence. Thus it clearly appeared with the advance of precision in the measuring instruments, that, for instance, to characterize isotropic linear elastic materials two constants were needed and not only one as suggested by the molecular model (relation 7.43).[86] The greater the accuracy and reliability of the experimental results the more the theoretical predictions of Cauchy and Poisson were disclaimed, though it was not clear why [534].[87]

A first attempt to adapt the 'classical' molecular model to the experimental results consisted in relaxing some of the basic assumptions. Poisson was among the first, in a memoir read before the Académie des sciences de Paris in 1829 [291], to formulate the hypothesis of non-puntiform molecules and crystalline arrangement; the idea of central forces depending only on the mutual distance between (the centers of) the molecules was thus released.

[85] p. 209.

[86] See the results found by Guillaume Wertheim (1815–1861) [346, pp. 581–610]. For bibliographical references to experiments on the constitutive relationships also see [464].

[87] pp. 481–503.

Cauchy also expressed doubts about the validity of the classic molecular model in some memoirs of 1839 [74][88] and in a review of March 1851 [74], of Wertheim's memoirs about the experimental determination of the elastic constants. Cauchy stated that the molecules in the crystalline bodies should not be considered as point-like but very small particles composed of atoms. Since in crystals there is a regular arrangement of molecules, the elastic moduli are periodic functions of spatial variables; assertions taken later by Saint Venant [253].[89] In order to obtain a constitutive relation with uniform coefficients, Cauchy expanded the number of elastic moduli, finally reaching only two in the case of isotropic materials.[90]

Gabriel Lamé in his works on the theory of elasticity [218, 220] raised a number of questions on the issue. For example, much of the XX lesson of the *Leçons sur les coordonnées curvilignes et leurs diverses applications* of 1859 [220] was dedicated to the concerns about the real nature of the molecules, to the assumption about the exact mutual actions, to what is a reasonable form of the law of the intermolecular actions, what is the direction of the latter.

Although the results of the molecular theory of elasticity were clearly considered unsatisfactory even by the followers of the French school of mechanics, it was not the case for the validity of the molecular approach. One of the main proponents of this approach was Saint Venant; his ideas on the matter, as well as in publications to his name, are contained in the enormous amount of notes, comments and appendices to the *Theorie der Elasticität fester Körper* of Alfred Clebsch (1833–1872), translated into French [92], and to the *Résumé des leçons donnés a l'école des pontes et chaussées* of Navier [253] where Saint Venant said:

> The elasticity of solid bodies, as well as of fluids, […], all their mechanical properties prove that the molecules, or the last particles composing them, exert on each other actions [which are] repulsive [and] infinitely growing for the smallest possible mutual distances, and becoming attractive for considerable distances, but relatively inappreciable when such distances, of which they [the molecular actions] are functions, assume a sensible magnitude [250].[91] (A.7.27)

For crystal bodies the classical molecular model seemed not to be valid:

> I do not yet refuses to recognize that the molecules whose various settings make up the texture of the solids and whose small change of distance produce noticeable strains called ∂, g are not the *atoms* constituent the matter, but are unknown groups. I accordingly recognize, thinking that the actions between atoms are governed by laws of intensity depending on the distances only where they operate, it is not certain that the *resultant* actions and that of the molecules must exactly follow the same law of the distances of their centers of gravity. We also consider that the groups, changing distances, can change orientation [92].[92] (A.7.28)

[88] s. 2, vol. XI, pp. 11–27, 51–74, 134–172.

[89] Appendix V, p. 689.

[90] A detailed reconstruction of Cauchy's topics is shown in the Appendix V, written by Saint Venant [253, pp. 691–706].

[91] pp. 542–543. My translation.

[92] p. 759. My translation.

But, added Saint Venant, this is only an ideal situation, because the ordinary bodies are not crystals, also the thermal motions produce a chaotic situation that on average leads to a law of action at a distance of the molecules substantially of the same type as that which there is between the *atoms*. Saint Venant made the six components of the tension to depend linearly on the six strain components, yet resulting in an elastic relationship in terms of 36 coefficients. However he continued to admit the validity of the relations of Cauchy and Poisson, which for isotropic bodies, leads to a single constant:

> The thirty-six coefficients […] reduce to two [...] and one may even say to one only […] in the same way as the thirty six coefficients are reducible to fifteen [253].[93] (A.7.29)

Saint Venant knew very well that these conclusions were contradicted by experiments, and since he did not find evident defects in the molecular theory of elasticity, preferred to accept that there are no isotropic bodies in nature:

> Yet experiences […] and the simple consideration on the way cooling and solidification take place in bodies, prove that isotropy is quite rare […]. So, instead of using, in place of the equations […] with one coefficient only, the formulas […] with two coefficients […], which hold, like these others, only for perfectly isotropic bodies, it will be convenient to use as many times as possible the formulas […] relative to the more general case of different elasticity in two or three directions [253].[94] (A.7.30)

In some work on the *Journal de mathématiques pures et appliquèes*, from 1863 to 1868 [313–315],[95] Saint Venant introduced the concept of *amorphous bodies* (amorphes corps) to define the properties acquired by the bodies initially isotropic as a result of geological processes. In this state, the mechanical properties are characterized by three coefficients and not just two as is the case for the isotropic bodies.

Saint Venant contributed more than 200 pages of notes and appendices to Navier's lessons in order to present experimental results and attempts to explain the paradox, showing a wide knowledge of the liter ature of his time (among others, he cited Savart, Wertheim, Hodgkinson, Regnault, Oersted, Green, Clebsch, Kirchhoff, Rankine, William Thomson). In the end, however, the question remained, because there was no agreement between the approaches of Saint Venant's contemporaries. Although it was clear that two elastic constants were necessary, where was the flaw in a theory attractive and apparently founded as Navier's, Cauchy's and Poisson's?

The debate between the scholars of mechanics was strengthened, from different points of view, by the works of Augustin Cauchy, George Green and Auguste Bravais (1811–1863), who gave life to different schools of elasticity.

7.4.2.2 Alternative Models of Matter

The molecular one was not the only model with which physicists and mathematicians tried to represent the behavior of elastic bodies. An alternative model characterized

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[93] p. 582. My translation.

[94] p. 583. My translation.

[95] In the order: pp. 353–430, pp. 297–350, pp. 242–254.

by a lower ontological commitment is represented by the mathematical deformable continuum which concerned mono-dimensional, two-dimensional or three-dimensional deformable bodies, thus replacing the rigid body model which in the past represented the prototype of a continuous body. The continuum is an idealization of reality which has the advantage of not requiring a priori assumptions about the structure of the matter and is easy to study.

The origins of continuum mechanics, may be searched far away; the first investigations about solid bodies in which there was consciousness to be working with a highly idealized model to which one can apply mathematical relationships suited to this idealization, can be found in the works of Jakob Bernoulli on the deformable beam, D'Alembert on the vibrating strings and Euler on the deformability and the critical load of beams, of Lagrange on the deformability of mono-dimensional elastic systems. But for what the modern continuum mechanics is concerned reference should be made to Cauchy's former works.

On September 30, 1822, one year after Navier's memoir, Cauchy presented to the Académie des sciences de Paris a memoir that dealt with the study of the elasticity according to a continuist approach, with a discussion largely unchanged since then. That of Cauchy was a purely phenomenological approach, in line with the empiricist tendencies that had developed near the French scientists.[96] The matter was modeled as a mathematical continuum without any assumptions of physical nature about it.

The fact that the continuous model does not make a priori assumptions on the constitutive relationships lets one tackle the static or dynamic problem in a more rational form. Indeed, one can write before the balance equations between tensions and external forces (including the inertial ones) and then impose the constitutive relationships. This division was clearly stated in Lamé's *Leçons sur la théorie mathématique de l'élasticité des corps solides* of 1852 and is classic today. The problem of continuum, consisting in the search at every point for the displacements and internal forces due to external actions, is divided into three phases.

1. In the first phase one writes the balance equations between internal and external forces (equilibrium).
2. In the second phase one writes the relations between strains and displacements (congruency).
3. In the third phase one writes the relationships between stresses and strains. The equations obtained can be joined to obtain the solution.

This division is less natural in the molecular model in which the constitutive relationship is partly implicit in the model itself.

With respect to stresses Cauchy turned to an analogy with the pressure of fluids. However differently from the case of fluids, stress is not necessarily orthogonal to the surfaces of separation of two parts of a body and is not necessarily a compression:

> If in an elastic or non-elastic solid body a small invariable volume element terminated by any of the faces is made [imagined] rigid, this small element will experience on its different

[96] For a discussion of the empiricist conceptions of French science in the first half of the XIX century, see [602].

> sides, and at each point of each of them, a determined pressure or tension. This pressure or tension is similar to the pressure a fluid exerts against a part of the envelope of a solid body, with the only difference that the pressure exerted by a fluid at rest, against the surface of a solid body, is directed perpendicularly to the surface inwards from the outside, and in each point independent of the inclination of the surface relative to the coordinated plans, while the pressure or tension exerted at a given point of a solid body against a very small element of surface through the point can be directed perpendicularly or obliquely to the surface, sometimes from outside to inside, if there is condensation, sometimes from within outwards, if there is expansion, and it can depend on the inclination of the surface with respect to the plans in question [75].[97] (A.7.31)

This statement sets aside any constitutive assumptions on the matter, but relies on the concept, then still not fully accepted, of distributed force. The relations between internal forces and deformations, i.e. the constitutive relationships, may have all general nature and the number of elastic constants that define the problem is simply determined by a count of the components of the stress and strain. In its most complete version Cauchy's continuum model leads to stress-strain relationship defined by 36 coefficients.

In his work *Sur la condensation et la dilatation des corps solides* of 1827 Cauchy defined the strains introducing the local deformation of the linear element as a percentage change in the length of an infinitesimal segment belonging to a continuum [76]. It, in the context of small displacements, came to depend on the six functions $\partial\xi/\partial a$, $\partial\eta/\partial b$, $\partial\zeta/\partial c$, $\partial\xi/\partial b + \partial\eta/\partial a$, $\partial\xi/\partial c + \partial\zeta/\partial a$, $\partial\eta/\partial c + \partial\zeta/\partial b$, also found in the analysis of the deformation of the molecular model, which assume again the role of strain components. Cauchy gave a geometric meaning only to the first three components, which represent the variations of unit length in the direction of the coordinate axes. In this he was less explicit than Euler and Lagrange who, in the study of the statics and dynamics of fluids, introducing linearized strain, had given the geometric meaning of angular distortion to the other three components [212].[98]

In a major work of 1841 Cauchy introduced the local finite and infinitesimal rotation of a segment in a given direction, and the average value in all the directions [83]. The linear elastic constitutive law had been introduced in 1828 [81]. In the initial part of the memoir, Cauchy consistently with his summary of 1823, assumed a single constant of proportionality between the principal stresses and strains. Soon after, in the same memoir, Cauchy introduced the constitutive law by means of two constants. The use of two elastic constants implies that to characterize the intermolecular forces as proportional to the displacement of the molecules is not equivalent to consider the voltages proportional to the strains term by term.

A different approach was that of George Green (1793–1841), who in a work of 1839 [172] also followed a phenomenal point of view assuming a three-dimensional continuum to model matter, uninterested even in the concept of internal forces. Green, however, turned to a mechanical principle, that of the existence of a potential of the internal forces, which somehow gave some theoretical force to his arguments. He

[97] p. 300. My translation.

[98] pp. 208–209. See also [450], pp. 288–292; 332–334].

spoke of the theory of elasticity in his work of 1839 [172] where he studied the propagation of light.

> Cauchy seems to have been the first who saw fully the utility of applying to the Theory of Light those formulae which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces supposing always that in the mutual action of any two particles, the particles may be regarded as points animated by forces directed along the right line which joins them. *This last supposition, if applied to those compound particles, at least, which are separable by mechanical division, seems rather restrictive; as many phenomena, those of crystallization for instance, seem to indicate certain polarities in these particles* [emphasis added]. If, however, this were not the case, we are so perfectly ignorant of the mode of action of the elements of the luminiferous ether on each other, that it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature; more especially if this principle include in itself as a particular case, those before used by M. Cauchy and others, and also lead to a much more simple process of calculation. The principle selected as the basis of the reasoning contained in the following paper is this: In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But, this function being known, we can immediately apply the general method given in the Mécanique Analytique [...] [172].[99]

Green considered a function of the components of the strains called "potential function" ϕ, whose exact differential gives the sum of the forces multiplied by the elementary displacement. If the strains are very small ϕ can be developed in a "very convergent" series:

$$\phi = \phi_0 + \phi_1 + \phi_2 + \text{etc.} \tag{7.44}$$

where $\phi_0,\ \phi_1,\ \phi_2$ are respectively homogeneous functions of degree $0, 1, 2, \ldots$ of the six components of the strain, each function being "very great" when compared to that of higher order [172].[100] One can neglect ϕ_0 (an immaterial constant) and ϕ_1 (the undeformed configuration is assumed equilibrated and for the principle of virtual works it is $\delta\phi = \phi_1 = 0$). Neglecting the terms of order higher than the second, the potential function is represented in each point of the body by ϕ_2, which, as a quadratic form of six variables, is completely defined by 21 coefficients. For isotropic bodies Green found again two constants.

Saint Venant rejected Green's approach because it lacked a mechanical basis, especially in relation to the concept of force. While Cauchy allowed it a moderate ontological commitment, and when it was more comfortable he treated matter as a continuous medium, Saint Venant consistently supported the molecular model because, according to him, the forces could only be explained by the interaction between mass points. Saint Venant's conceptions of mechanics are well summarised in his *Principes de mécanique fondé sur la cinématique* of 1851 [312]; for him all matter is made of non-extended molecules, and mechanics is simply the science

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[99] p. 245.
[100] p. 249.

through which one determines the distances of certain points from other points, at a given instant, knowing what these distances have been at other instants. These are the main principles he assumed at the foundation of mechanics:

1. In a system of two molecules only, they undergo equal and opposite accelerations along the line joining them, with an intensity depending on their distance only.
2. In a system made of several molecules, the acceleration of a given molecule is the geometrical sum of the accelerations it would acquire if it were subjected separately to the actions of each of the other atoms (the rule of parallelogram).
3. The mass of a body is a number proportional to the number of molecules that can be supposed it contains.
4. The force is nothing but the product of the mass by the acceleration [312].[101]

Saint-Venant thanked God, not Newton, for the simplicity of these assumptions: "God not only wanted invariable laws, he also wanted accelerations to depend only on distance. Further, he wanted superposition."[102]

While Green, for whom the hypothesis of intermolecular forces opposing along the line joining the molecules was too restrictive and, given the complete ignorance of the 'real' law of action, thought that one must use a weaker criterion, Saint Venant contested both the rejection of the principle of action and reaction, the fundamental law of mechanics, and the choice of a quadratic function to approximate the potential, because, according to him, without any physical hypothesis there is no reason to assert that an arbitrary function should have dominating quadratic terms:

> If the scientific prudence prescribes to not rely on any assumption it does equally prescribe to hold under strong suspect what is clearly contrary to the great synthesis of the generality of facts [...]. Also we reject any theoretical formula in formal contradiction with the law of action as a continuous functions of the distances of the mass points and direct according to the lines connecting them in pairs. If, using this formula, it is easier to explain certain facts, we always look it as an *expedient* too convenient [...] [253].[103] (A.7.32)

The question of the correctness of the adoption of one or two constants for linear and isotropic elastic bodies remained open for a long time in the mechanics of the XIX century; the studies of Lamé and Saint Venant could not reconcile the corpuscular approach of Navier, Cauchy and Poisson with that of Green. On the other hand, at least until the second half of the century the precision drawn from experimental research was limited and some indecision was allowed, but when the experimental results became more reliable, the hypothesis of two constants prevailed, without one being able to explain where and why the corpuscular theory fell into fault.

In 1866 a monograph on crystallography was published by Auguste Bravais [60] at first sight not directly connected with the study of the mechanics of deformable bodies of the French and English schools. In fact, the relevance of the monograph was not so much its approach and the crystallographic classification, which constituted

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[101] p. 64.

[102] From un unpublished manuscript quoted in [424, p. 331].

[103] p. 747. My translation.

its heart, but rather the premises, that were essential for overcoming the impasse in the choice of the number of elastic constants.

Bravais believed, based on his studies of crystallography, that the crystalline materials could be considered as a set of molecules, in the limit reduced to their center of gravity, but with the fundamental assumption that these molecules also have their own orientation in space, repeatable in a regular lattice in the construction of matter (an assumption already suggested by Poisson).

Matter is therefore due to aggregates of regular lattices whose components are no longer, as for Navier, Cauchy and Poisson simple mass points, but points with orientation. A modern mechanician would say that the microscopic descriptors of the model have a local structure, that characteristic of a rigid corpuscle. The molecules of the crystalline bodies are small polyhedra, the vertices of which are the centers of the forces that each molecule of the body exchanges with the contiguous ones:

> The molecules of crystallized bodies will be from now on polyhedra whose vertices, distributed in any way around the center of gravity, will be the centers, or poles, of the forces emanating from the molecule [60].[104] (A.7.33)

This view of the matter dictated the way for the German scientist Woldemar Voigt's (1850–1919) model, which will put an end to the search for the answer about the rightness of taking one or two elastic constants for homogeneous and isotropic linear elastic materials [390].

7.4.2.3 Elastic Structures

Development of the theory of elasticity by engineers was due to their attempts to improve the comprehension of the structural elements of civil and industrial engineering that were beams, trusses, plates and shells. But the studies of these elements could be carried out independently of any general theory of elasticity. The need to introduce the deformation in the calculation of the structures occurred when it was realized that the statics of rigid bodies was not enough to study those structures today, called statically indeterminate, or structures subject to redundant constraints [386].

The formulation and the first solution of the problem is usually traced back to Euler in 1773 [139] who wanted to calculate the 'pressure' originated by a rigid body placed on a horizontal plane with more than three supports. Euler declared the problem not to be soluble by the known laws of statics. An important theoretical question then arose: can the equations of statics be used appropriately to solve contact problems? And in this case are the equations sufficient or must one formulate some new principle? The problem assumed a great importance in Italy and its history is well traced by Isaac Todhunter (1820–1884) and Karl Pearson (1857–1936), who tend to trivialize it [654].[105]

[104] p. 196. My translation.

[105] vol. II/1, p. 411.

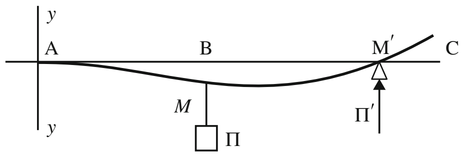


Fig. 7.4 Clamped and supported beam [254, Fig. 48, Pl. II]

Among the attempts to solve the problem of redundant supports, the first success on both theoretical and practical aspects was Navier's. He in lessons of 1824 published in 1826,[106] dealt with the case of a plane beam with a number of external constraints greater than three, which cannot be solved with the equations of statics. Navier recognized that a solution could be obtained only if one accepted the deformability of the beam:

> When a rigid rod loaded by a weight is supported on a number of support greater than two points, the efforts that each of these points of support must endure are undetermined between certain limits. These limits can always be determined by the principles of statics. But if the rod is assumed to be elastic, the uncertainty ceases entirely. We consider here only one of the questions of this kind, the simplest that can be proposed [254].[107] (A.7.34)

To understand the originality and the limits of Navier's approach, consider the simple case of the beam AMM' of Fig. 7.4 clamped at one end, simply supported at the other end and loaded by the weight Π at the intermediate point M. The beam has one constraint more than these strictly necessary to avoid rigid motion.

Without many comments Navier replaced the support in M′ with a vertical force Π' assuming that its value is sufficient to maintain M immovable. From the theory of elastic beams developed by Jakob Bernoulli and Euler, Navier was able to evaluate the displacement of the point M′ due to the forces Π and Π' separately. Assuming Π' as an unknown and imposing that the vertical displacement of M′ be zero, Navier obtained a linear equation in Π'. Once Π' is known the beam can be supposed loaded by known forces and solved by means of the equations of statics.

Navier's approach is today classified as a *method of forces*, in which the constraint forces are determined by imposing the congruence equations (the respect of constraints). It is likely that Navier did not recognize the method in its generality, because he limited himself to solving only restrained beams fixed by external constraints. Moreover, he did not dispose of a general method for the calculation of the displacements for structures of arbitrary shape. Saint Venant attributed the merit of having extended Navier's approach to each type of structure, at least from the theoretical point of view [253][108] and in 1843 [310] he outlined very clearly the approach of the method of forces:

> This method is to get the displacements of the points of the parts leaving as indeterminate the intensities, the lever arm and the directions of the forces we are talking about. Once the displacements are expressed as functions of the sought quantities, the conditions they must

[106] According to Saint Venant already in 1819 Navier considered the case [253, p. cviii].

[107] p. 241. My translation.

[108] p. ccxii.

> meet are imposed at the points of support or clamping of the various parts, or at the connection points of the various parts in which a piece must be divided, because the displacements are expressed by different equations. In this way, one gets to have as many equations as unknowns, because *obviously* [emphasis added] there is, in matters of mechanical physics, no indetermination at all [310].[109] (A.7.35)

Saint Venant applied his methodology for the analysis of statically indeterminate structures in two other memoirs of the same year (1843) [310, 311]. In any case he was not able to outline a simple procedure, although it probably would have been enough for him to deepen the calculation of displacements in the beam. As a matter of fact the engineers of the time were not able to calculate even simple statically indeterminate structures such as trusses and frames with welded nodes that were beginning to be used in construction.

A satisfactory success, at least from a practical point of view, was reached a few years later thanks to Henry Bertot (fl 1850s) and Benoît Paul Emile Clapeyron (1799–1880) who arrived at a general solution for simple and continuous beams with many supports, in the form that today is called the equation of the three moments [87].[110] The theory of structure reached a quite final form in the second half of the XIX century, thanks mainly to the Italian and German schools of engineering [386, 399].

7.5 Hydraulic Machines

The study of the development of fluid and thermal machines offers an interesting point of view to analyze the interactions between science and technology and allows to appreciate the influence on the technology of the birth of applied mechanics. In this section I will consider the behavior of hydraulic machines only, ignoring the important sectors of wind machines, and postponing the analysis of thermal machines to another section.

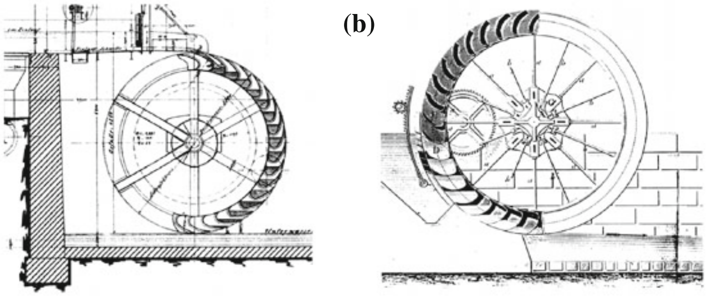
Hydraulic machines can be of different kinds; for a synthetic description see [392, 635, 639, 644]; here reference is made to waterwheels only, for their widespread diffusion. There exist mainly two types of waterwheels: the undershot waterwheels and the overshot waterwheels, whose difference is shown in Fig. 7.5a–c. There is also another interesting wheel, intermediate between the two, the breast wheel in which water enters from the mid points—or breast —of the wheel (Fig. 7.5b).

In the undershot machines, water flows beneath the wheel and hits blades or paddles evenly diffused around the periphery of the wheel. They are moved by the impulsion of the particles of water. In the overshot machines water was led above the wheel and instead of blades often there are buckets which are filled with water and move the wheel by means of the gravity of the water in the buckets.

[109] p. 953. My translation.

[110] p. 1077.

(a) (b)



(c)

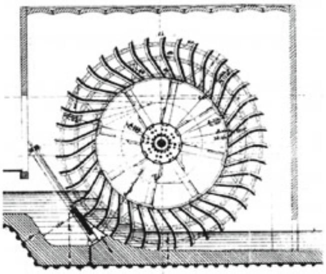


Fig. 7.5 Types of water wheels: **a** overshot wheel; **b** breast wheel; **c** undershot wheel [587, p. 453]

7.5.1 Old Hydraulic Machines

The waterwheel steadily evolved since its introduction, to pump water and mill grain. It is not clear where it had its origin; it however rapidly spread out as described by Roman, Greek and Chinese sources. There is evidence that the familiar vertical waterwheel developed within the Roman Empire and rapidly spread out [429]. Overshot wheels required a large head (2–10 m), therefore they were usually confined to hilly areas, or required extensive and expensive auxiliary constructions. On the other hand, undershot wheels could operate with a small head (0.5–2 m), hence they could be located on small streams in flat areas, near to population centers. It is widely considered that the most dramatic industrial consequences of waterwheels occurred in the Middle Ages, when the scale of milling considerably increased with the development of large towns. From grinding wheat and pumping water in antiquity, water powered mills evolved to forge iron, full cloths, saw woods and stones, and to metalworking and leather tanning [429].[111]

In the XVIII century the waterwheels received new attention because of the rising of the manufacturing industry and its increasing need for energy. Before the introduction of steam the only way to get energy from nature was by means of the motion of

[111] p. 194.

water (and air). However the available streams of water were limited and an increase of energy could come only by improving their efficiency. The problem of efficiency of water-wheels and their history in the XVIII century has been the object of rather recent studies [392, 393, 612], which also make general considerations on the role of the hydraulic energy in society. The purpose of the present section is to clarify some misunderstandings and also to present some reflections on the interaction between science and technology in this particular field.

Hydrodynamics at the beginning of the XVIII century was scarcely developed. Only the theoretical results reported by Newton in the second book of his masterpiece, *Philosophiae naturalis principia mathematica*, [264], by Torricelli in his *Opera geometrica* [333][112] and the experimental analysis of Edme Mariotte in his *Traité des eaux et autres corps fluides* [204] were of some help. In this situation scientists could consider very simplified models only. Besides, engineers or at least some of them, were no longer practical men; they knew hydraulics quite enough and, mainly, had a scientific attitude toward experiments which were carried out using models of reduced size and accurate measurements. There were thus elements for science and technique to cooperate. Scientists were the first to be involved, but the results they found were useless from a practical point of view because far from the actual findings. For this reason the development of the hydraulic machines in the whole XVIII century was greatly influenced by engineers that experimented different kinds of wheels, in particular overshot wheels and wheels with curved blades.

The difference between theory and practice was hardly accepted by the scientific community, thus the need to interpret the experimental results was pressing. But it took nearly a century from the first theoretical analysis, to reach a satisfactory interpretation of the hydraulic phenomena and to suggest a way to build more efficient machines thanks to the studies of French engineers, especially Poncelet [298, 299].

A first attempt to evaluate the efficiency of waterwheels was carried out by Edme Mariotte who measured the force of a water stream by means of a counterbalancing weight, drawing the conclusion that the force varies as the square of the velocity of impact [240],[113] a result which was then provided by Newton on theoretical basis [268].[114] A sophisticated analysis on the undershot wheels was soon given by Antoine Parent in 1704 [286].[115]

Antoine Parent's theoretical analysis of the undershot wheels
In his memoir *Sur la plus grande perfection possible des machines* [286] Parent considered the idealized system of wheels, deprived of any friction, of Fig. 7.6. A fluid (water) flows through a channel from left to right; it spins the large wheel CBD as a consequence of the force exerted by the fluid in B. The rotatory motion of FGH is transmitted by means of teeth to the small wheel HMR, around whose axis a rope wraps and lifts a weight. The fluid flows through the channel EB with

[112] p. 265.
[113] p. 205.
[114] Part II, Theorem 27.
[115] pp. 116–123, 323–338.

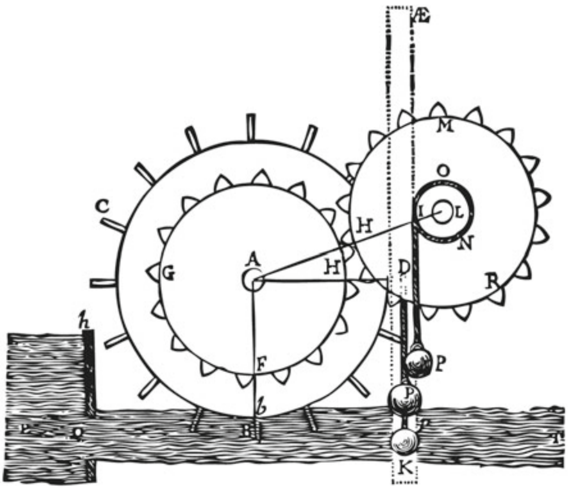


Fig. 7.6 Parent's undershot wheel (Redrawn from [286, p. 326])

uniform velocity V; the larger wheel CBD rotates with constant angular velocity and the velocity at point B of the immersed blade is equal to x, so that the relative velocity of the water with respect to the blade in B is $V - x$. Parent wanted to calculate the weight p that the smaller wheel HMR is able to lift with velocity u [286].

He made the following assumptions, the last two of them, though fundamental, were not made explicit:

(a) The force exerted by the water flow on the blade in B is proportional to the square of the relative velocity between the blade and the water.
(b) A form of the principle of virtual work can be applied—as the friction is negligible—assuming that a steady state is reached in which the forces are balanced and all goes as in an equilibrium situation.[116]
(c) Only one blade at a time was considered immersed in water.
(d) The stream was considered to be perpendicular to the blade.

Parent indicated with P the force exerted by the fluid in B with the blade at rest (then with a relative speed of the fluid equal to the absolute velocity V); and called *natural effect* (effect naturelle) of the fluid the product PV [286].[117] The product pu of the lifted weight p by its velocity u is the (general) *effect* of the fluid.

Assume with Parent: B = AB, b = AH; C = LH; c = LI, Q the weight necessary to equilibrate P.[118] The relation between P and Q is given by the principle of virtual

[116] Parent introduced his principle of virtual work in a not problematic way, as a well established one, in the form: "The speed of B is always to that of P in the compound proportion of the radius AB to the radius AH and the radius LH to the radius LI" [286, p. 330]. However he spent some interesting words to justify the application of this static principle to the case of motion, referring also to Galileo.

[117] p. 326.

[118] Note that in his calculations Parent used the same symbol P to indicate both the suspended weight which equilibrates the wheel and the force exerted on the blade. I prefer to differentiate the

work as:

$$Q = \frac{\mathrm{BC}}{bc} P, \tag{7.45}$$

where BC/bc measures the ratio of the virtual displacements of the blade B (horizontal) and that of the weight Q (vertical).

When the wheel CBD rotates, instead of P which is proportional to V^2, the force on the blade in B is P^*, lower than P and proportional to $(V - x)^2$, being $V - x$ the velocity of the water relative to the wheel. The weight p (to be equilibrated with P^*), and the weight Q (equilibrated with P) are in the same proportion of P^* and P, and so:

$$\frac{P}{P^*} = \frac{Q}{p} = \frac{V^2}{(V - x)^2}, \tag{7.46}$$

which is an equation between x and p. Parent solved it with respect to x, which gives:

$$x = V \frac{\sqrt{Q} - \sqrt{p}}{\sqrt{Q}}. \tag{7.47}$$

With simple kinematical considerations, the velocity u of the weight p is obtained:

$$u = x \frac{bc}{\mathrm{BC}} = V \frac{\sqrt{Q} - \sqrt{p}}{\sqrt{Q}} \frac{bc}{\mathrm{BC}}. \tag{7.48}$$

By multiplying u by p the effect of the machines is thus:

$$pu = V \frac{\sqrt{Q} - \sqrt{p}}{\sqrt{Q}} \frac{bc}{\mathrm{BC}} p. \tag{7.49}$$

If the geometry of machine (bc and BC) and the absolute velocity V of the fluid are kept as constant, the effect of the machine only depends on p. Parent found the maximum value of the effect with the use of the *Calculus*, for $p = 4/9Q$:

> Art. V. If one now assumes B, C, b, c as constants and p is decreased, or decreased as far as possible, that is to say, we do it through all changes in size which is possible, the value that makes the machine to produce its greatest effect, there will be p variable in the general values of the effect of the preceding article, and taking the differential of the value, namely, $\left(\sqrt{P} - \frac{2}{3}\sqrt{p} \times \frac{Vbc}{BC\sqrt{P}} dp\right)$ with the purpose to equate it to zero (according to the method

(Footnote 118 continued)
symbols, retaining P for the force exerted on the blade and indicating the suspended equilibrating weight as Q.

of the infinitesimals) it results the equality ($\sqrt{P} = \frac{3}{2}\sqrt{p}$), from which ($\frac{2}{3}\sqrt{P} = \sqrt{p}$), and finally ($\frac{4}{9}P = p$) [286].[119] (A.7.36)

From the value $p = 4/9Q$ which makes maximum the efficacy, the maximum value of the effect can be obtained simply by substituting this value of p in the Eq. (7.49), also considering the equilibrium relation $Q = P\ \mathrm{BC}/bc$, giving:

$$pu = \frac{4}{27}PV, \tag{7.50}$$

thus the effect of the fluid is $4/27$ of the natural effect. The optimal value of velocity can be obtained from (7.47), resulting in $x = V/3$. Notice that all these values are independent of the geometry of the machine and the velocity of fluid.

If one wants to evaluate the efficiency of the machine, namely the ratio between the work made in a second and the available living force (or in modern term potential energy)—as was done by many scientists and engineers of the XVIII century, such as Smeaton, Daniel Bernoulli for instance—he has to rework the Eq. (7.50). In the following I refer to an application made by Parent in the case of water falling from a height H [286].[120] He assumed for the force exerted on the paddle of the wheel at rest the value associated to the static pressure of the water, $P = \gamma \mathrm{HA}$ being A the section of the vein and γ the specific weight of the fluid. Replacing the value of P in the Eq. (7.50) gives (the symbols are adapted, because Parent's symbols conclict among themselves):

$$pu = \frac{4}{27}\gamma HAV = \frac{4}{27}qH, \tag{7.51}$$

where q is the flow, weight in a second.

Assuming the static value for P appears as an incongruence to a modern reader, because if Parent, as it seems from his reasoning, was considering a wheel immersed in a river then he should assume for P the dynamic value; if instead he was assuming that the wheel was placed in a channel having the same width of the blade, P is correctly evaluated by the static value, but the dynamic analysis leading to the Eq. (7.50) is not tenable. But Parent could not have made this confusion. Most probably he followed the knowledge of the time, based on experiments, that before Daniel Bernoulli's *Hydrodynamica* of 1738 seemed to indicate at first glance that the pressure of the aqueous stream flowing uniformly is equal to the weight of an aqueous cylinder, the base of which is the orifice through which the water flows, and the height of which is equal to the height of the water above the orifice, an idea also suggested by Newton's first edition of the *Principia*. According to Bernoulli "to this thinking the majority, in fact all, adhered and do adhere up to this time, because it agrees wonderfully with other experiments also, especially those which are customarily performed on

[119] p. 331. My translation. In this quotation P corresponds to my Q.
[120] p. 333.

spheres moved in a resisting medium" [37].[121] If Parent had used the correct value suggested by Bernoulli for the force, $P = 2\gamma HA$, he would have found a double effect $pu = 8/27qH$.

Apart from the last consideration that is not central, it can be said that Parent's approach is elegant and with no errors; its limitations are due to the idealization of the model. His results can be regained at ease using modern notations and concepts. To this purpose see [429], where also the analysis of the overshot wheel is reported. Although Parent's analysis was idealised, its results were adopted by many scientists of the XVIII century such as John Theophilus Desaguliers (1683–1744) [170],[122] Colin Maclaurin (1698–1746) [319][123] and Leonhard Euler (1707–1783) included [135].[124]

Jean-Charles de Borda's theoretical analysis of the undershot wheels

Jean Charles de Borda (1733–1799), in the *Memoire sur les roues hydrauliques* of 1767 [53], much later than Parent, when hydrodynamics had already become a quite mature science, reconsidered the problem of the efficiency of water wheels. He studied several situations. Besides the classical undershot wheel with plane blades he also studied a wheel with curved blades and an overshot wheel. Here the first case is referred to, while the latter is discussed in one of the following sections. The case of the wheel with curved blades is not discussed because it was too difficult a problem for de Borda and was satisfactorily solved only in the XIX century [95, 96, 298, 299].

De Borda derived the behavior for the undershot wheel with plane blades starting from the analysis of the wheel having a vertical axis. Since a detailed presentation of de Borda's results would be too long, they will be summarized and adapted to Parent's problem and symbols, also considering that de Borda's text contains many misprints. Moreover as he used two different approaches, one based on the principle of living force the other based on D'Alembert's principle [102],[125] which however give the same result, for the same reason of economy the former only is described; for the latter approach refer to [382].

The hydrodynamic context considered by de Borda is different from Parent's; while Parent assumed the force on the blade resulting from the friction in a large flow of water, a river for instance, de Borda assumed an impact of the water on the blade moving in a narrow channel as large as the blade. The water enters with a speed V and after a complex interaction with the paddles of the wheel, more than one at a time, lives with a speed x, having so lost in the impact the velocity $V - x$. The following expression of the effect, as a balance between initial and lost living forces

[121] p. 289.
[122] vol. 1, p. 434–435
[123] pp. 452–455.
[124] p. 189.
[125] pp. 73–75.

can be written [53][126]:

$$pu = qH - \frac{1}{2}\frac{q}{g}\left[(V-x)^2 + x^2\right], \tag{7.52}$$

where q is the flow of the fluid (weight for unit of time), g the acceleration of gravity, H the height of the fall of water necessary to reach the speed V. In this equation qH represents the available initial living force, $1/2qx^2$ the living force lost because the water leaves the wheel with non-zero speed x, $1/2q(V-x)^2$ is the living force lost by the water in the impact with the paddle, which is proportional to the square of the lost velocity $(V-x)$. This last result, now known as Carnot's theorem, is to be attributed to de Borda himself.[127]

The maximum effect pu is obtained by finding the maximum of the right side of the Eq. (7.52) considered as a function of x, which is attained for $x = 1/2V$; to which the maximum value of pu is associated:

$$pu = qH - \frac{1}{4}\frac{V^2}{g} = \frac{1}{2}qH. \tag{7.53}$$

where Torricelli's theorem is accounted for ($V^2 = 2gH$). Thus the efficiency of the undershot machine would be 1/2, that is a much higher value (about twice) than that found by Parent. Notice that Eq. (7.53) is mine, obtained completing de Borda's reasoning; however somewhere in his memoir he explicitly said that the theoretical maximum efficiency of the undershot machine is 1/2 [53].[128] Adding that in practice this result is never reached, de Borda stated that the lower value 3/8 should be assumed [53].[129]

In a comment de Borda tried to justify his result which is different from Parent's:

> What my solution says is contrary to what has been said so far by the mathematicians who worked on the matter who all found that to produce the greatest impact on a paddle wheel, it should be left to the paddles one third of the velocity of the fluid that hits them, and here I show what this result is based on. It is considered but one paddle on this wheel AB, against which the force is sought of the shock of the fluid; it was found by calling B the velocity of the fluid and V that of the paddle, that the shock was proportional to $(B-V)^2$ and as the effect of the impeller is necessarily proportional to the speed of the blades multiplied by the shock of the fluid, the effect of the wheel was given by $V(B-V)^2$, from which it is obtained for the maximum $V = 1/3B$. But it was observed that the movement in question,

[126] p. 282. The symbol adopted here and in the following, for the sake of comparison, are Parent's; de Borda's equation reads actually as: $(pu =)EH - E/g\left[(u-V)^2 + V^2\right]$, in which E is the flow and u and V stay respectively for V and x.

[127] Navier in his *Détails historiques sur lémploi du principe des forces vives dans la théorie des machines et sur diverses roues hydrauliques* of 1818 reported that de Borda corrected Daniel Bernoulli's assumption, according to which the loss of living force in an impact of fluid was proportional to $v^2 - v'^2$, stating that it was instead proportional to $(v-v')^2$, being v and v' the speed before and after the impact [255, p. 149].

[128] p. 284.

[129] p. 285.

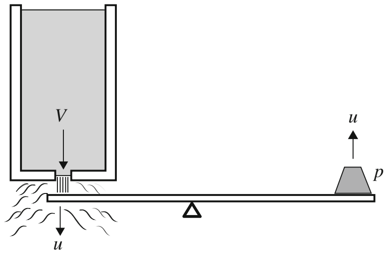


Fig. 7.7 Daniel Bernoulli's model to evaluate the efficiency of an impacted wheel (Simplified drawing from [286, Table VIII, Fig. 54])

> the action of the water is not exerted against an isolated blade, but against several blades at a time, and that these blades closing all the breading of the small canal and removing from the fluid the velocity that this has more than that, the amount lost by the fluid, and therefore the shock experienced by the paddle movement is no longer proportional to the square of the difference in fluid velocities and pallets, but only to the difference in the speed; from which it follows that the effect is represented by $V(B-V)$[Parent's symbol $x(V-x)$], and not by $V(B-V)^2$ [Parent's symbol $(V-x)^2$]; now matching $V(B-V)$ to a *maximum*, we find $V = 1/2B$ [Parent's symbol $x = 1/2V$] [53].[130] (A.7.37)

It is sometimes argued in papers in the history of science that Parent made calculation errors [393, 612] and De Borda would instead have found exact results correcting the error due to the approximation in considering a wheel at the time and a factor two, which Parent had neglected:

> In 1767 Borda published a short paper correcting the two main errors of Parent and harmonising theory with experiment [393].[131]

Parent's theory is actually correct if properly understood. The difference with de Borda depends on the different hydraulic context assumed by the two scientists.

Johann Albrecht Euler—a son of Leonhard—in a memoir submitted in 1754 for a prize competition, which he actually won, analyzed separately undershot, gravity and reaction wheels [129]. For the undershot wheel he found Parent's result, that is an optimal speed for the paddles $1/3$ of the speed of water and an efficiency equal to 8/27 qH [129].[132] In his *Hydrodynamica* of 1738 Daniel Bernoulli [38] had reinterpreted Parent's result. The situation to which Bernoulli referred is illustrated in Fig 7.7, where a flow of water moving with speed V hits an arm of a lever moving with speed u and consequently raising a weight p. Bernoulli found that the maximum value of the product pu (Parent's effect) is obtained for $u = 1/3V$, and the maximum value is given by $4/27PV$. Bernoulli also remarked that the small efficiency of the undershot wheels had to be ascribed to the fact that part of the water living force was lost to keep still the high speed of the water flowing after the impact against the paddles of the wheel had occurred [38].[133]

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130 pp. 273–274. My translation.

131 p. 212.

132 p. 12.

133 pp. 193–195.

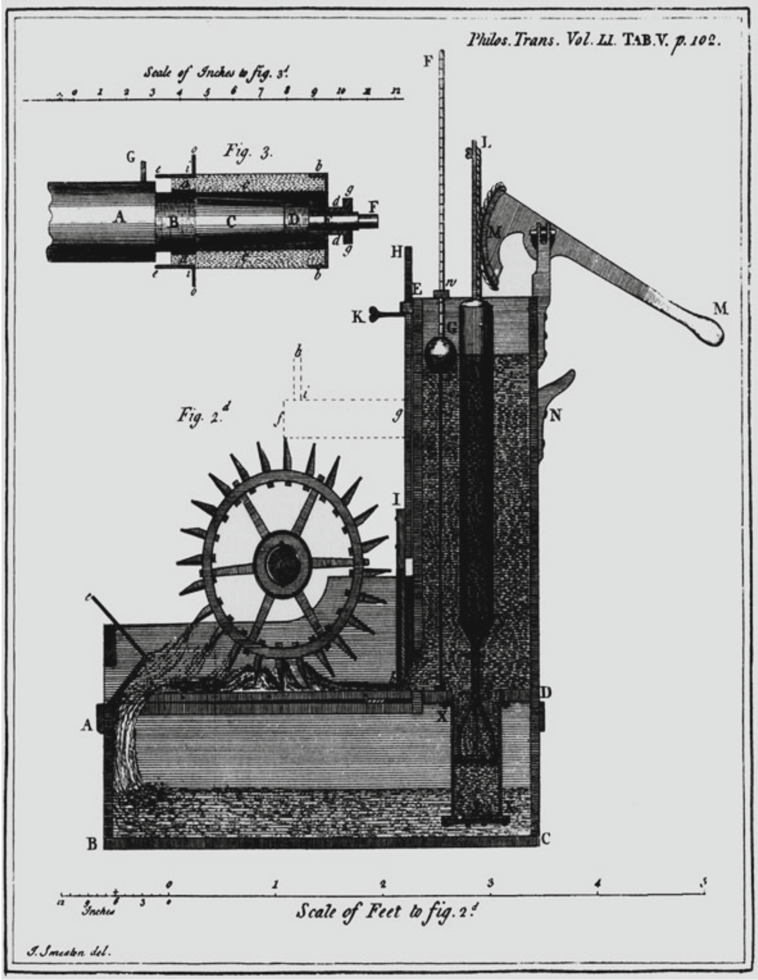

Plate 7.1 Smeaton's experimental set [318, p. 102] (reproduced with permission of Biblioteca Guido Castelnuovo, Università La Sapienza, Rome)

John Smeaton's experimental investigations on undershot wheels
The first systematic experiments on waterwheels were probably those of the English engineer John Smeaton (1724–1792) who in 1759 published *An experimental enquiry concerning the natural powers of water and wind to turn mills, and other machines, depending on a circular motion*, before de Borda's memoir. Here he compared under and overshot wheels [318].

Smeaton's attention to water wheels was due to the demand of English industry for an improvement of the efficiency of existing water wheels. Being not convinced by Parent's results he performed numerous experiments on the model shown in Plate 7.1, where ABCD is a reservoir which collects water for recirculation after its

action on the waterwheel. Water is pumped out of the waterwheel via a hand pump (MN is the handle of the pump, L the pump rod) to another higher reservoir DE. The water in DE was maintained at a constant level by observing the graduated rod FG, while the water released on the wheel was controlled by the rod HI. A rope connected to the axle of the wheel in O and led through the pulleys P and Q raised a pan of weights, R, used for measuring the wheel's output (not apparent in figure). The apparatus could be adapted to test overshot wheels as shown by the dotted line in the cross-sectional view.

Smeaton defined the *original power* of the water as the product between the quantity of water released in a given time and the height that water comes down from. The *effect* of the machine is the sum of the weight raised by the action of this water and the weight necessary to overcome the friction, multiplied by the height the weight will be raised to in a given time. The efficiency is the ratio between effect and original power [318].[134] In one of his experiments where the power was 3,970 pounds $\times$ inches in a minute (the product of the flow of 264.7 lb of water multiplied by the height of fall of 15 in.), by varying the raised weight, he found that the maximum effect corresponded to 1,266 pounds $\times$ inches in a minute (the product of a weight of 9.375 lb raised to a height of 135 in.), for an efficiency of $1{,}266/3{,}970 = 32\,\%$, greater than that provided by Parent (25 %) but lower than that provided by de Borda (50 %). The ratio between the velocity of the blades of the wheel and the velocity of water was often greater than that foreseen by Parent, arriving in some cases close to 1/2 instead of 1/3. Also the weight raised was much greater, (3/4) instead of 4/9 of the equilibrating weight [318].[135] Smeaton justified the difference between theory and experiment as a consequence of different assumptions:

> It must be remembered, therefore, that, in the present case, the wheel was not placed in an open river, where the natural current, after it has communicated its impulse to the float, has room on all sides to escape, as the theory supposes; but in a conduit or rate, to which the float being adapted, the water cannot otherwise escape than by moving along with the wheel. It is observable, that a wheel working in this manner, as soon as the water meets the float, receiving a sudden check, it rises up against the float, like a wave against a fixed object; insomuch that when the sheet of water is not a quarter of an inch thick before it meets the float, yet this sheet will act upon the whole surface of a float, whose height is 3 in.; and consequently was the float no higher than the thickness of the sheet of water, as the theory also supposes, a great part of the force would have been 10ft, by the water's dashing over the float [318].[136]

In a subsequent paper Smeaton summarized his experimental results, rhetorically exaggerating the difference between experimental and theoretical findings, asserting also that for a large wheel (as the wheels of actual mills), the efficiency is greater arriving up to 50 %:

> For if that conclusion were true, only 4/27 of the water expended could be raised back again to the height of the reservoir from which it had descended, exclusively of all kinds of

[134] pp. 106–107.
[135] p. 115.
[136] pp. 113–114.

friction, &c. which would make the actual quantity raised back again still less; that is, less than one-seventh of the whole; whereas it appears from table I of the said volume [Smeaton 1759], that in some of the experiments here related, even upon the small scale on which they were tried, the work done was equivalent to the raising back again about one quarter of the water expended; and in large works the effect is still greater, approaching towards half, which seems to be the limit for the undershot mills, as the whole would be the limit for the overshot mills [emphasis added].
[…]
The velocity also of the wheel, which according to M. Parent's determination, adopted by Desaguliers and Maclaurin, ought to be no more than one-third of that of the water, varies at the maximum in the above mentioned experiments of table, between one third and one half but in all the cases there related, in which the most work is performed in proportion to the water expended and which approach the nearest to the circumstances of great works, when properly executed the maximum lies much nearer to one half than one third [319].[137]

Antoine Deparcieux's and John Smeaton's experiments on overshot wheels

The overshot waterwheel received no attention by scientists probably because there was the spread opinion that they had the same efficiency as the undershot ones [612].[138] This was the opinion of Leonhard Euler also, who in a his work of 1754 denied that the overshot wheel had any advantage over the undershot ones [135].[139] Bernard Forest de Belidor [31][140] maintained that an undershot wheel is six times more efficient that an overshot one, while Desaguliers on the contrary affirmed that a "well-made overshot mill" may be ten times more efficient than an undershot wheel [120].[141]

The first known study on the overshot wheels was that of Antoine Deparcieux (1703–1768) who is usually classified as an engineer though a member of the Académie des sciences de Paris. The interest of Deparcieux derived from the desire of Madame de Pompadour to have current water from a small river, the Blaise in Crésy, raised to a height of 50 m. Because of the small flow of the river, an undershot wheel would not have been able to satisfy the request. Deparcieux was brought to think that the efficiency of the overshot wheels should be higher than that of the current undershot wheels by assimilating the water, that descends and works as an engine and the water that should be raised, to two weights which are located on two opposite sides of a pulley and are connected by a rope.

I soon saw that I could get a much better use of water weight, considering it as weights which falling raise others: but how has one to take the wheel [119].[142] (A.7.38)

He stated to have made experiments with a pulley using as power a weight of 96 ounces which raised weights of 24, 32, 40, etc. ounces registering the amount these weights rise in one second, that is the velocity in the first part of motion. The velocity ranged from 85 in./s for a weight of 24 ounces to 20 in./s for a weight of 72

137 pp. 456–457.

138 p. 274.

139 p. 198.

140 vol. 1, p. 286.

141 p. 532.

142 p. 607. My translation.

ounces [119].[143] On the basis of his results Deparcieux suggested a simple thought experiment by imagining two waterwheels equal to each other but with their buckets inclined in opposite directions. The wheel receiving the falling water was able to raise water in the other wheel under the condition that the raised water was less than the falling one. And, in the same way as in a pulley if the wheels rotate very slowly, the amount of raised water will be equal to that fallen, and the efficiency of the overshot waterwheel should reach 100 %.

Deparcieux's explanation, on the greater efficiency of wheels that rotate slowly, actually has no weight. The experiment of the pulley is of course truthful, but here accelerated motions are concerned. In the case of the waterwheel there is instead a stationary motion. In this situation it can be shown that the velocity of the wheel, at least ideally, has no influence on the efficiency. The greater efficiency, usually registered for the overshot wheels that rotate slowly, depends on the construction methods and operation. In [429] the reasons for the efficiency of the overshot wheels to decrease with the increasing of speed are illustrated.

With his apparatus, Smeaton was able to experiment on an overshot wheel with water flowing from the tape indicated with fg in Plate 7.1. He found that using the same wheel with plane blades, the efficiency was double that of the undershot wheels and confirmed the results obtained by Deparcieux, that the efficiency of the wheel increased by slowing its speed.

Smeaton was convinced that most of the difference between over and under wheels were due to the loss of living force of the water in the latter case associated to its change in shape during impact. He also proposed an (unsatisfactory) explanation for the increase of efficiency of the overshot wheel by slowing the speed of the wheel, assuming that at the higher speed the efficiency of the water pressure was lower.

> When the velocity is greater [water] does not press so much upon the bucket as when it is less, the power of the water to produce effects will be greater in the less velocity than in the greater: and hence we are led to this general rule, that, *caeteris paribus*, the less the velocity of the wheel, the greater will be the effect thereof [318].[144]

In subsequent years Smeaton performed many experiments on the impact of non-elastic bodies assuming that the loss of living force in the impact was due to a change of shape of the bodies. The following quotations summarizes Smeaton's ideas about the energy (modern term) required to change the shape of a body:

> To obviate this, those of the old opinion seriously set about proving, that the bodies might change their figure, without any loss of motion in either of the striking bodies.
> […].
> On the other hand, if it can be shown that the figure of a body can be changed, without a power, then, by the same law, we might be able to make a forge hammer work upon a mass of soft iron, without any other power than that necessary to overcome the friction resistance, and original vis inertiae, of the parts of the machine to be put in motion: for, as no progressive motion is given the mass of iron by the hammer (it being supported by the anvil). no power call be expended that way; and if none is lost to the hammer from changing the figure of the

[143] p. 609.

[144] p. 133.

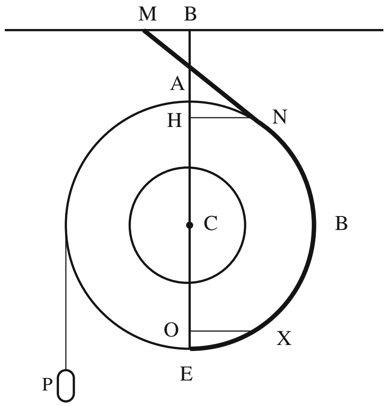

Fig. 7.8 de Borda's wheel (Redrawn from [53, p. 286])

iron, which is the only effect produced, then the whole power must reside in the hammer, and it would jump back again, to the place from which it fell, just in the same manner as if it fell upon a body perfectly elastic, upon which, if it did fall, the case would really happen: the power, therefore to work the hammer would be the same whether, it fell upon an elastic or non-elastic body; an idea so very contrary to all experience [320].[145]

Thanks to Smeaton, the overshot wheels reached a high efficiency and contrasted the success of new-born steam machines.

However much Mr. Smeaton's valuable observations may have been disregarded by authors, they have not been lost to practical men [...] [As a result of his experiments] he determined to apply the water, in all cases, so that it should act more by its weight, and less by its impulse; and the advantage gained by that improved construction was found to be fully equal to his expectation. It was afterwards so generally adopted and improved upon by himself and by other engineers in this country, that although undershot water-wheels were, about fifty years ago, the most prevalent, they are now rarely to be met with; and wherever economy of power is an object, no new ones are made [612].[146]

Jean-Charles de Borda's theoretical analysis of overshot wheels
Smeaton, at least in 1759, did not know de Borda's analysis of the overshot wheels which, though correct, needed to be interpreted. De Borda, considering a very idealized wheel whose buckets did not leave water, drew the conclusion that an overshot wheel as in Fig. 7.8, where the stream of water MN is tangent to the wheel, reaches its maximum effect when $\mathrm{BH} = 0$ and the wheel rotates with zero velocity, confirming that the efficiency of this kind of wheel increases by lowering the speed of rotation. The balance of 'energies' is that expressed by the Eq. (7.52), where now V is the speed of the water after the descent $h = \mathrm{BH}$, given by $\sqrt{2gh}$; H is the height of fall HE, x is the speed with which the water leaves the wheel corresponding to the tangential speed of the buckets. So that the effect pu assumes the expression [53][147]:

[145] pp. 342–343.

[146] p. 29 f1.

[147] p. 281.

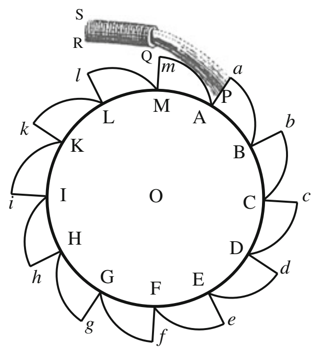

Fig. 7.9 Euler's gravity wheel (Redrawn from [129, Table II, Fig. 9])

$$pu = qH - \frac{1}{2}\frac{q}{g}\left[(\sqrt{2gh} - x)^2 + x^2\right], \tag{7.54}$$

which reaches its maximum value for $h = x = 0$:

$$pu = qH, \tag{7.55}$$

which indicates an efficiency of 100 %.

At the end of his paper [53][148] De Borda noticed that the efficiency is, in practice, substantially independent of x, as its change with x, is rather small.

De Borda's was preceded by Johann Albrecht Euler, who in his already mentioned memoir of 1754 in the study of the efficiency of the gravity wheel, illustrated in Fig. 7.9, concluded that if the buckets were large enough to collect all the water of the stream and if the diameter of the wheel was equal to the height of the fall, the efficiency of the gravity wheel would be 100 % [129]. Euler's work was however scarcely known; it was not quoted neither by de Borda nor by Smeaton.

7.5.2 New Hydraulic Machines

In the XVIII century hydrodynamics was developing thanks to the theoretical work of Leonhard and Johann Euler, D'Alembert and Daniel Bernoulli [37, 101, 136, 137] who had also made possible a theoretical investigation of the operation of water wheels. But their works on the subject, published in scientific papers or books, had a very limited impact on the technological development of the wheels, mainly because they were not read by engineers. In particular, they were not read by Smeaton and were soon forgotten. Thus the legacy of experiments and the theoretical speculations of the XVIII century left to the XIX century consisted mainly of the two points concerning the optimization of the efficiency of water wheels:

(a) The impact of the water upon the paddles of the wheel should be avoided.

[148] p. 286.

(b) The wheel must move so that the water is unloaded with the minimum possible speed.

These conclusions were collected by Lazare Carnot [69] who based his theory of machines on the conservation of living force and impact for insensible degrees.

Thus if one had to summarize in a few words the role of science in the technology of water-wheels in the XVIII century, he would be tempted to say that it was modest, almost negligible as claimed by [509, 612]. In my opinion there was instead a fruitful interaction between science and technology. In fact, though the application of rational mechanics based on a high formalization had a limited impact, on the contrary, less formalized theoretical considerations, such as those of Parent and de Borda, had a decisive role, despite their high degree of idealization. The hydrodynamical studies aimed at assessing the thrust of fluids also had decisive importance. Moreover, considering Deparcieux's and Smeaton's peers as foreign to science, as was done by some historians asserting the low influence of science on technology, is certainly debatable and not shared by all. For example [588] considered Smeaton's contribution as an example of the direct application of science to technology

In the XIX century the development of hydraulic machines was brought in the frame of applied mechanics, where theory, the rational mechanics, and practice, experiments in the laboratory and in the field, were carried out by the same people, the modern engineer, determining a great improvement in the efficiency of all kind of machines. A prevalent role was played by the military engineers of the École de applications de l'Artillerie et du Génie, in particular Jean Victor Poncelet and Arthur Jules Morin (1795–1880) [358].

These engineers were deeply involved in mathematics and physics to consider themselves more as scientists than as practitioners; for instance they addressed their memoirs to the Académie des science instead of to technological journals. They made great recourse to experiments, but not so much to verify the goodness of the general mechanical theories behind their designs. The experiments had rather two main scopes. On the one hand to highlight some minor defects of the machines to be corrected after a theoretical review of the problem; on the other hand to evaluate numerical values of some correcting coefficients which allowed one to pass from theoretical to practical formulas. This was due not to errors in theory but to simplified assumptions. For example very often the conservation of living forces—or the work—was assumed and friction was not modeled distinctly; its effect was taken into account when performing experiments under various operating conditions and arranging tables of correcting coefficients.

After some preliminary works [357] Poncelet prepared the *Mémoire sur les roues verticales à palettes courbes mues par en dessous, suivi d'expériences sur les effets mécaniques de ces roues* concerning the undershot waterwheels. It was presented before the Académies des science in 1824 and published in the Annales de chimie et physique in 1825, with minor revisions [298]; an improved version was published in 1827 [299].

Poncelet's purpose was to satisfy Carnot's (and de Borda's) requirements for an efficient machine: avoiding the loss of living force by impact and the releasing of

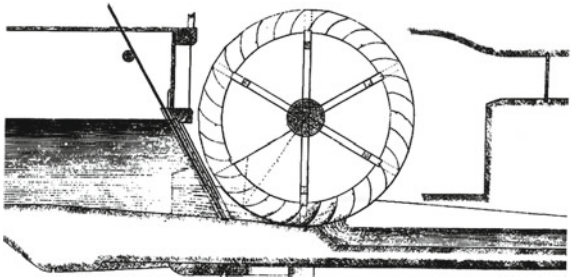

Fig. 7.10 Poncelet's undershot wheel (Adapted from [299, Planche 1])

water with significant speed. He reached his scope by assuming curved and inclined blades as shown in Fig. 7.10; probably not a new idea, but a good idea that was pursued with due firmness:

> The idea to substitute curved blades to plane blades of the old systems seemed so natural and simple that one can think that its arose to everyone; so I did not attribute a great merit to it. But because the simplest ideas are often those which found the most difficulties to be accepted, I did not want limit myself to theoretical speculations [298].[149] (A.7.39)

With these devices the undershot wheels could reach, at least theoretically, an efficiency of 100 %. The water wheel as proposed by Poncelet is now known as a *Poncelet wheel*.

Later studies and experiments highlighted some weakness of Poncelet wheels [357], which however spread and for a long time were competing with the water turbines introduced by Benoit Fourneyron (1802–1867), to replace the waterwheel, around 1830 [147]. Contrary to what was commonly believed, turbines did not fully replace the waterwheels and their design was in the syllabus of engineering faculties at least until 1940. They disappeared only after the second world war. Today new attention is paid to waterwheels properly designed, both undershot and overshot, as an economical solution to get energy from water streams with low head [587].

7.6 The Emergence of Thermodynamics

7.6.1 Conservation of Energy

The history of the law of conservation of energy has been the subject of careful studies by historians of science; among the most significant accounts, because of the importance of their authors, are those of Mach [565] and Kuhn [547]. The history of the conservation of energy, as those of all the fundamental concepts of science, has a fundamental difficulty. Searching into the past for something that looks like a modern concept leads to two kind of possible errors. On the one hand one can see the origin of this concept in scientists who used words similar to modern ones referring

[149] pp. 144–145. My translation.

to different concepts; on the other hand one cannot see it in other scientists, who albeit with a different language, used concepts similar to the modern ones.

Of course Mach and Kuhn are conscious of the problem and try to solve it with an accurate analysis of texts, without however being completely convincing. This depends, not on their incapacity as historians of science but from the fact that the problem is ill posed. To make the situation even more difficult there is the fact that the term *energy*, as that of force, is still today endowed with a great ambiguity both semantic and ontological. And in many modern formulations of mechanics the concepts of energy and force are reduced to functional relations among quantities endowed with a clear physical meaning such as displacement, velocity, etc. For this very reason for example it can be said that the statement of the living force as espoused by Lagrange, which is a theorem derived from the equations of motion, a first integral of them, is closer to the modern principle of the conservation of mechanical energy than the statement espoused by Helmholtz, for instance, where the energy is something of substantial.

For this reason, rather than searching for the origin of a modern concept, it is better, without ambiguity, to try to reconstruct the way the scientists of the period faced the problems that today are considered in some way related to what is known as the law/principle/theorem of conservation of energy (modern meaning). For the sake of space here reference is made only to those researches that were considered more relevant by the scientists of the times (of by the modern ones also I must add).

Wilson Scott in his precious *The conflict between atomism and conservation theory* [629], proposes that the areas to investigate are essentially four:

1. The long metaphysical tradition of indestructibility of force advocated by Leibniz. This was sometimes interpreted erroneously. Most notable in supporting this point of view was Hermann von Helmholtz in a famous paper *Uber die Erhaltung der Kraft* read before the Physical society of Berlin in 1847.
2. Calculations dealing with mechanical equivalent of heat based on specific heats at constant pressure and at constant volume, by Sadi Carnot in 1824, Karl Holtzmann in 1845, Julius Robert Mayer in 1842, and James Prescott Joule in 1845, together with supplementary evidence for the new view by Mark Séguin and Karl Frierich Mohr, Gustave Adolphe Hirn.
3. The experimental observation of losses of force together with the theoretical explanation of these losses in terms of conversion; namely, those arising in the inelastic collision of soft bodies (conversion to work—William Wollaston and Peter Ewart, end of the XVIII century), Jean Victor Poncelet in 1820s; those arising from impact of hard aqueous bodies (conversion to heat—Joule); from the mechanical force in a magnetic field (conversion to electrical current—Michael Faraday); from the resistance to electrical flow in a metal wire (conversion to heat—Joule).
4. The precise measurements of conversion afforded by electro-chemical experimentation—Faraday's electro-chemical equivalent in voltaic cells and electrolysis—and the porous-plug experiment [629].[150]

[150] p. 254.

A similar suggestion comes from Kuhn also who sees three meaningful factors in developing the idea of conservation:

(a) The availability of the processes of conversion.
(b) The interest for machines.
(c) The spread of the Naturphilosophie [549].

In his work Kuhn evidences that various researchers moved independently one of the other, having even difficulty in appreciating the analogies of different achievements, that could be seen in the second half of the 1800 s only. The process of 'discovery' of the law of conservation of energy would thus have been all but straightforward.

In the XVIII century, with Bernoullis' works the metaphysical principle of the living force of Leibnizian origin (more or less the principle of conservation of mechanical energy), was applied to practical mechanics. The idea was very fertile because it allowed one to easily solve problems, such as those encountered in the study of the modern machines, otherwise unsolvable either with Newtonian mechanics of mass point or Eulerian mechanics of rigid body.

The Bernoullis attributed a high degree of realty to the living force considered to be like a substance and that was seen not only in the body in motion but also, as latent living force, stored for example in elastic springs. With Lazare Carnot and the French school, the quantity which was monitored, and of which the conservation was stated, was work. Carnot proved the conservation of work when the relative displacement of impacting bodies varies for insensible degree, starting from his principles of mechanics. His was then a theorem, but his successors moved to consider work as a substance. The living force, inverting Bernoulli's point of view, was considered as potential work, because a body in motion can make work. In this case too the existence of a law of conservation of mechanical energy could be assumed.

At the beginning, the application of this principle seemed to be non-problematic. Of course the principle of conservation of living forces or of work did hold in no situation because of frictions, always present. But this was not considered a problem by mechanicians used to study highly idealized models. There were other more fundamental difficulties, however, which undermined the principle. It was well known that in the impact among non-elastic bodies, in particular the so-called hard bodies, the living force is lost; accordingly the principle of conservation cannot be maintained, neither as an approximate assumption. To this difficulty that could partially be circumscribed for solid bodies, one should add the losses concerning fluids, the study of which became fundamental in the turn of the XIX century, because the fluids were to move the machines. D'Alembert and Lagrange were convinced that for fluids the principle of conservation of living force was valid, because the impact of the particles of water occurred for insensible degrees. Soon, however, it was realized that this was not true in many practical situations. From this point of view works by de Borda and Smeaton, illustrated in the preceding sections, were fundamental.

Carnot, Poncelet and especially Coriolis perfectly knew these problems. Their reaction was of a pragmatic kind: it was necessary to avoid losses. However, at a certain moment the problem of understanding the reason for these losses became urgent. Leibniz and Bernoulli already had given their response: there is no loss; the

living force apparently lost, for example in a plastic impact, is actually acquired by the small particles of the bodies in a chaotic motion, not apparent as global motion. Smeaton had suggested that in the case of impact a part of living force is spent to produce a change in shape of the impacting bodies, and this held true for fluids too.

But these observations, though ingenious, remained generic and all considered scarcely interesting for many scientists of the period, because a quantitative and independent evaluation to be compared with the loss of living force, was missing. At a certain stage the idea that the lost living force is transformed in heat emerged. However for this idea to get space it needed to wait for a better understanding of the nature of heat, The first theories about heat were mechanists, whose root could be found in the Greek atomists, in Francis Bacon, Newton, etc.,[151] but they were too generic. In the XVIII century the most diffuse theory of heat was that of caloric. It had two versions; in the first version, the oldest one, the caloric was assumed as a thin fluid which permeates bodies. The temperature of a body depends on the concentration of caloric; when this fluid is plentiful it is expelled from the body giving rise to radiant heat. In the second version it is assumed the existence of an infinitely subtle substance, the ether. When a body is heated its particles are put into motion; this motion is then transferred to the particles of ether that can assume rotatory and translational motions. The particles of ether are responsible for the transmission of heat through the body and to the other bodies [629].[152]

7.6.1.1 Joule's Experimental View

The theory of caloric was useful to explain many phenomena among which that of cooling of gas which undergoes an adiabatic expansion: the cooling should derive from rarefaction of caloric [546]. This theory could not however explain the heat production due to friction, exposed for the first time with a wealth of details by Thompson Benjamin Rumford (1753–1814), who published his results in 1798 [309]. Rumford in his experiments proved that friction could produce heat and to that purpose performed some quantitative measurements. The most interesting experiment was that classified as third in Rumford's paper, the result of which was commented by him with the following words:

> At the end of two hours, reckoning from the beginning of the experiment, the temperature of water was found to be raised to 178 F^0. At 2 hours 20 minutes it was at 200^0; and at 2 hours 30 minutes it ACTUALLY BOILED! [309].[153]

Rumford noted that the heat actually generated by friction with the work expended by two horses and accumulated into two hours and half, might be made to boil 25 pounds of ice-cold water [309].[154] From these data with simple calculations it can

[151] For an interesting account see [52], part. II, *The history of fire*, pp. 221, 222.

[152] pp. 217–218.

[153] p. 92.

[154] p. 96.

be found that the heat required to raise a pound of water 1 F^0 was equivalent to the work of about 1,000 pound-foot, a value much higher than the actual value of about 770 pound-foot. The discrepancy in the result is justified by the loss of heat in the cannon and in the atmosphere not accounted for [195].[155]

Rumford's experiments were interpreted by most as the empirical verification of the mechanical theory of heat, for which thermal phenomena were explained by means of some form of motion of the elementary particles; the faster the motion the higher the temperature. It must be said, indeed, that the mechanical theory of heat—at least for a modern—is a sufficient condition to justify the complete transformation of work into heat and then the conservation of mechanical and thermal energy. But it is not a necessary condition, and the transformation of mechanical in to thermal energy can be justified otherwise, though not by caloric:

> One must convince himself that the motional concept of heat is inessential as is its conception as a substance. Both ideas were favored as impeded by accidental historical circumstances [565].[156]

And indeed when the conservation of energy was also extended to electromagnetic and chemical phenomena it became difficult to speak about the transformation of mechanical work into the other form of energies.

A major contribution to the affirmation of the mechanical theory of heat came from James Prescott Joule (1818–1889). In the 1840s he performed many experiments on conversion of different kinds of energies (modern term) into heat. Particularly interesting and famous are those regarding the conversion of mechanical force (the word used by Joule for work) into heat [191–195].[157] Below I will comment with some details the 1849 work which is the most comprehensive.

Joule started with quotations by Locke and Leibniz to point out his adherence, at this date, to the mechanical theory of heat:

> Heat is a very brisk agitation of the insensible parts of the object, which produces in us that sensation from whence we denominate the object hot; so what in our sensation is heat, in the object is nothing but motion. Locke.
>
> The force of a moving body is proportional to the square of its velocity, or to the height to which it would rise against gravity. Leibnitz [195].[158]

Then he summarized the state of art giving space to Rumford's experiment and referring to Davy's and Dulong's works related to the transformation of mechanical work into heat. He also quoted Mayer's paper of 1842 [245] by asserting he was the first to say that heat can be generated by fluid friction. Notice that generation of heat from agitation of fluids was then considered as a completely different phenomenon than that of generating heat by friction, and also doubtful.

Joule proved the conversion of mechanical force into heat by employing a paddle wheel to agitate two different fluids: water and mercury and made tests also for the

[155] pp. 61–62.

[156] pp. 42–43.

[157] For a comment about the possible priority of Sadi Carnot over Joule see [629, pp. 243–244].

[158] p. 61.

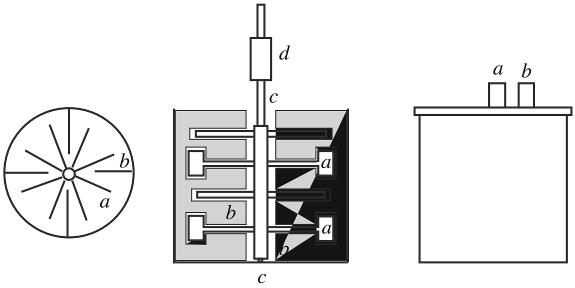

Fig. 7.11 Details of Joule's apparatus for producing friction (Redrawn from [195, p. 64])

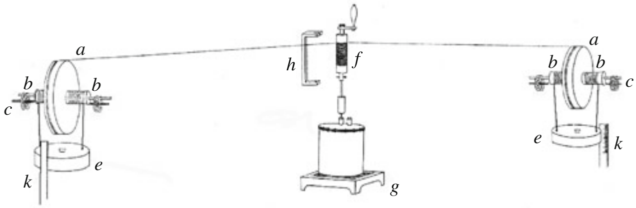

Fig. 7.12 Perspective view of Joule's apparatus (Adapted from [195, p. 64])

friction of cast iron. For the sake of simplicity only the case of water is referred to. Figure 7.11 represents the apparatus employed for producing the friction, consisting of a brass paddle-wheel furnished with eight sets of revolving arms, *a*, *a*, &c., working between four sets of stationary vanes *b*, *b*, &c, affixed to a framework also in sheet brass. The brass axis of the paddlewheel worked freely, but without shaking, on its bearings at *c*; at *d* it was divided into two parts by a piece of boxwood intervening, so as to prevent the conduction of heat in that direction. The right part of Fig. 7.11 represents the copper vessel into which the revolving apparatus was firmly fitted: it had a copper, the flange of which, furnished with a very thin washer of leather saturated with lead, could be screwed perfectly water-tight to the flange of the copper vessel. In the lid there were two necks, *a*, *b*, the former for the axis to revolve in without touching, the latter for the insertion of the thermometer.

Figure 7.12 is a perspective view of the machinery employed to set the frictional apparatus just described in motion. *a*, *a* are wooden pulleys, one foot and 2 in. in diameter having wooden rollers 2 in. in diameter, and steel axles *cc*, *cc*, one quarter of an inch in diameter. The pulleys were turned perfectly true and equal to one another. Their axles were supported by brass friction wheels, the steel axles of which worked in holes drilled into brass plates attached to a very strong wooden framework firmly fixed into the walls of the apartment. The leaden weights *e*, which in some of the ensuing experiments weighed about 29 lbs, and in others about 10 lbs, a piece, were suspended by string from the rollers *bb*, *bb* and a fine twine attached to the pulleys *a* connected them with the central roller *f*, which could with facility raise the weights *e*. The wooden stool *g*, upon which the frictional apparatus stood, was perforated by

a number of transverse slits, so cut out that only a very few of wood came in contact with the metal, whilst the air had free access to almost every part of it. In this way the conduction of heat to the substance of the stool was avoided.

The method of experimenting was simply as follows: The temperature of the frictional apparatus having been ascertained and the weights wound up with the assistance of the stand h, the roller was reaffixed to the axis. The precise height of the weights above the ground having then been determined by means of the graduated slips of wood k, k, the roller was set at liberty and allowed to involve until the weights reached the flagged floor of the laboratory, after accomplishing a fall of about 63 in. The roller was then removed to the stand, the weights wound up again, and the friction renewed. After this had been repeated twenty times, the experiment was concluded with another observation of the temperature of the apparatus. The mean temperature of the laboratory was determined by observations made at the commencement, middle and termination of each experiment [195].[159]

Joule took a great care to evaluate the heat transferred by radiation, that absorbed by the apparatus and that absorbed by water. He found that the mean value of heat after 40 measurements was of 1 F^0 in 7.8423 pounds of water. The mean value of the mechanical force performed by the falling weight was 6,067.114 pound-foot, which gives (6,067.114/7.8423) 773.64 pound-foot for 1 F^0 in a pound of water [195].[160]

Joule's paper ended with a summary of the results he found and a general comment on the convertibility of mechanical force into heat:

> The following Table contains a summary of the equivalents derived from the experiments above detailed. In its fourth column I have supplied the results with the correction necessary to reduce them to a vacuum.
>
> TABLE IX.

No.	Material employed	Equivalent in air	Equivalent in vacuo	Mean
1	Water	773.640	772.692	772.692
2	Mercury	773.762	772.814	
3	Mercury	776.303	775.352	774.083
4	Cast iron	776.997	776.045	
5	Cast iron	774.880	773.930	774.987

> It is highly probable that the equivalent from cast iron was somewhat increased by the abrasion of particles of the metal during friction, which could not occur without the absorption of a certain quantity of force in overcoming the attraction of cohesion. But since the quantity abraded was not considerable enough to be weighed after the experiments were completed, the error from this source cannot be of much moment. I consider that 772.692, the equivalent derived from the friction of water, is the most correct, both on account of the number of experiments tried, and the great capacity of the apparatus for heat. And since, even in the

[159] pp. 63–66.

[160] p. 70. In the international unit of measure 773.64 pound-foot corresponds to about 4,160 J/Kcal. The presently accepted value is 4,185 J/Kcal.

> friction of fluids, it was impossible entirely to avoid vibration and the production of a slight sound, it is probable that the above number is slightly in excess. I will therefore conclude by considering it as demonstrated by the experiments contained in this paper.
>
> 1st. That the quantity of heat produced the friction of bodies, whether solid or liquid, is always proportional to the quantity of force expended. And,
>
> 2nd. That the quantity of heat capable of increasing the temperature of a pound of water (weighed in vacuo, and taken at between 55° and 60°) by 1°FHAR, requires for its evolution the expenditure of a mechanical force represented by the fall of 772 lbs. through the space of one foot [195].[161]

Joule, in other occasions, made also statements on the inverse process, i.e. that of transformation of heat into 'mechanical force':

> You see, therefore, that living force may be converted into heat, and that heat may be converted into living force, or its equivalent attraction through space. All, three, therefore—namely, heat, living force, and attraction through space (to which I might also add light, were it consistent with the scope of the present lecture)—are mutually convertible into one another. In these conversions nothing is ever lost. The same quantity of heat will always be converted into the same quantity of living force.
> [...]
> The knowledge of the equivalency of heat to mechanical power is of great value in solving a great number of interesting and important questions. In the case of the steam-engine, by ascertaining the quantity of heat produced by the combustion of coal, we can find out how much of it is converted into mechanical power, and thus come to a conclusion how far the steam-engine is susceptible of further improvements [196].[162]

Though the fact that work could be obtained from heat was largely testified by the existence of thermal machines, the precise process and measurement were not known. The first to make measurements of conversion from heat to mechanical work was Gustave Adolphe Hirn (1815–1890) [600]. Hirn did not make experiments on ad hoc experimental apparatus but used two existing steam machines. He measured the work produced and the heat disappeared in the expansion of steam and found the mechanical equivalence of heat with a process inverse to the usual. Experimental values furnished by Hirn gave contrasting values; more precisely he found different conversions from different kinds of machines. Though this did not disturb Hirn, who did not believe in the mechanical theory of heat, this disturbed Clausius who reinterpreted Hirn's results giving a mean value to the conversion of about 4,050 J/Kcal; not very far from results found by Joule (4,160 J/Kcal) [600],[163] [388].

7.6.1.2 Helmholtz's Metaphysical View

Many historians consider Helmholtz's paper *Über die Erhaltund der Kraft* of 1847 [178] as a fundamental step in the establishment of the principle of conservation of energy. This is a quite correct judgement, but one must be precise that Helmholtz's

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[161] p. 82.

[162] pp. 270–271.

[163] p. 247.

role was mainly that of the promotor of the idea than to propose new argumentation. His enterprise had success, at least in Germany, because of his great prestige as a scientist. The ideas of Helmholtz relative to conservation of energy and limited to his paper of 1847 are referred to below and will allow the reader to judge by himself on the matter.

Since the introduction, Helmholtz brought forth a position decidedly mechanistic based on a corpuscular conception of matter and the presence of various kinds of immutable forces between the corpuscles. The following quotation gives a clear idea of Helmholtz's mechanistic conception, which was intended to cover all physical phenomena:

> We have seen above that the phenomena of nature are to be attributed to immutable ultimate causes. This requirement can be expressed in the need to look for immutable forces in time, as causes. Matters with immutable forces (inexhaustible qualities) are those called (chemical) element in sciences. But if we imagine the universe is disassembled in elements endowed with immutable qualities, the only changes still possible in such a system, are spatial movements, that is consisting in motion and the external conditions through which the effect of the forces is modified, can only be spatial: the only forces are thus moving forces, their effect depending only on the spatial relationships [178].[164] (A.7.40)

It must be specified that Helmholtz's mechanistic view was not the standard one, as clear from the text. In the marginal notes to his work, in the 1881 edition, Helmholtz disputed the law of composition of force and the principle of action and reaction in many circumstances, connected mainly with electromagnetic phenomena [178].

Helmholtz's first step had to be to establish the principle of conservation of mechanical *force* (this is the term he used for energy). He with a little mathematics, in the case of central forces, which for him covered all cases, found again the old principle of living force, which for a system of mass points assumes the form [178][165]:

$$-\sum\left[\int_{rab}^{Rab}\varphi_{ab}dr_{ab}\right]=\sum\frac{1}{2}[m_a Q_a^2]-\sum\frac{1}{2}[m_b Q_b^2], \tag{7.56}$$

where φ_{ab} is the force between two elementary particles a and b located at the distance r_{ab}. The expression $\int_{rab}^{Rab}\varphi_{ab}\,dr_{ab}$ was named *force of tension* (Spannkräfte),[166] the term $\frac{1}{2}[m_a Q_a^2]$ living force (Lebendigenkräfte). These are Helmholtz's comments on the result he has found:

> Here we have again in the left side the sum of the forces of tension and in the right side the sum of the living forces of the whole system. We can now formulate the law: In all cases, the motion of free mass points under the influence of their attractive or repulsive forces whose intensities only depend on the distance, the quantity of the loss of tension forces is always

[164] p. 5. My translation.

[165] p. 14.

[166] In a section of his *Über die Erhaltund der Kraft* concerning magnetic forces between two bodies, Helmholtz qualified the integral $\int\varphi\,dr$ as the potential (Potential) of the two bodies [178, p. 45].

> equal to the gain in living force, and the income of the former, equal to the loss of the latter. Thus it is always constant the sum of the existing living and tension forces. In this general form, we can define our law as the principle of the conservation of energy [178].[167] (A.7.41)

Comparing the statement of Helmholtz with that of Daniel (and Johann) Bernoulli and the French engineers it should be said that there are not substantial differences. All think of conservation as something real. The merit of Helmholtz is therefore essentially to have reiterated and diffused the argument among the scientific community.

In the *Über die Erhaltund der Kraft* there are limited references to experimental situations, but there is a strong belief in the invariance of the force, and therefore in its preservation, which may have its origins in *Naturphilosophie*. The conservation of thermo-mechanical energy is presented as an *a priori* need that only subsequently is proved by the mechanical theory of heat adopted by Helmholtz, for which the heat was associated to translational and rotational motion of atoms, and also to their distortion—Helmholtz atoms were then composite bodies.

The experimental results Helmholtz referred to were those of Clapeyron and Holtzmann. Joule was referred to just to underline that he knew him; Mayer was not quoted at all. Helmholtz will right the wrongs toward Mayer, but never gave the right space to Joule; maybe for a nationalistic reason or simply because he did not understand the significance of Joule's work.

Before Helmholtz, but after the Bernoullis, expressions similar to *force of tension* were used without assigning them a physical meaning. For instance Lagrange, in the *Théorie de la libration de la Lune* and the *Mécanique analytique*, introduced the function $V = \int Pdp + Qdq + \cdots$, avoiding to give it a name, implying conservative force [208, 209].[168] Navier in 1821 used the expression *moment of a force* to indicate the quantity $1/2 f^2$, f being the distance between two molecules, proportional to the force exchanged [252].[169] Green and Gauss introduced the expression and the idea of potential to different fields; the former, in a paper of 1828 on static electricity used the expression *potential function* [173];[170] the latter in a paper on capillarity used the term *potential* only [253].[171] Green went back to potential in 1839 [172] justifying its existence on the impossibility of perpetual motion. The use of the potential function of molecular forces in the theory of elasticity occurred in the majority of European countries, excluding France.

With respect to the modern term *energy* it can be said it was used occasionally in many instances. It was used for example by Thomas Young:

> The term *energy* [emphasis added] may be applied, with great propriety, to the product of the mass or weight of a body, into the square of the number expressing velocity.[…] This product has been denominated the living or ascending force, since the height of the body's

[167] p. 14. My translation.

[168] p. 24; p. 36. Actually Lagrange used the symbol V in the *Théorie de la libration de la Lune* and the symbols Φ and Π respectively in the first and subsequent editions of the *Mécanique analytique*.

[169] p. 386.

[170] pp. 1–82.

[171] p. 784.

> vertical ascent is in proportion to it; and some have considered it as the true measure of the quantity of motion, but although this opinion has been universally rejected, yet the force thus estimated well deserves a distinct denomination [339].[172]

and by William John Macquorn Rankine (1820–1872):

> As the phrase 'potential energy', now so generally used by writer on physical arguments was first presented by myself in a paper *On the general law of the transformation of energy* [published 1855] [305].[173]

but the term spread thanks to William Thomson and Peter Guthrie Tait who in their textbook *Treatise on natural philosophy* [650] named *kinetic energy* the expression $1/2mv^2$, and Thomson who in his work of thermodynamics used t *mechanical energy* and *intrinsic energy* as the mechanical value of the effect the body would produce from the state in which it is given to the standard state [332].[174] Thomson was the first to prove on a thermodynamical basis the existence of the elastic potential energy for a linear elastic system which deformates isothermically [332]. In 1855 Thomson used *énergie potentielle* to distinguish it from *énergie actuelle* (kinetic energy) [332].[175]

7.6.2 Thermal Machines

There is knowledge of the application of steam to produce motion in the works of ancient scientists/engineers such as Ctesibus of Alexandria and Philo of Byzantium (third century BC). Many are the mechanical devices which one hears about Hero (first century AD) in the *Pneumatica* and the *Automata*. Of interest for the development of the steam engine is the well-known aeolipile [652].

To find other applications of steam one needs to arrive at the XVI century. Even then there were scientists, or in any case educated people and not practical men, who suggested thermal engines, based on some elementary concepts of pneumatics developed in the Renaissance, but no attempt was presented to quantify the effect produced. Among the first to exploit steam machines it is worth mentioning Giovanni Battista della Porta (1535–1615). He proved, on a laboratory scale, that steam could be used to move water, either by forcing it or by leaving it to be sucked up into a vacuum (caused by the condensation of steam). He considered the two aspects in different points of his *I tre libri de' spiritali* of 1606 [116]. Figure 7.13a shows an apparatus imagined by della Porta to raise water by means of steam. The steam generated in the boiler D is inflated into the recipient B filled with water. The steam pressure evacuates all the water in B through the tube C. Similarly Salomon de Caus (1576–1626) in his *Les raisons des forces mouvantes avec diverses machines* of 1615

[172] vol. 1, pp. 78–79.

[173] p. 229. For the *On the general law of the transformation of energy*, see [304].

[174] p. 57.

[175] p. 1197.

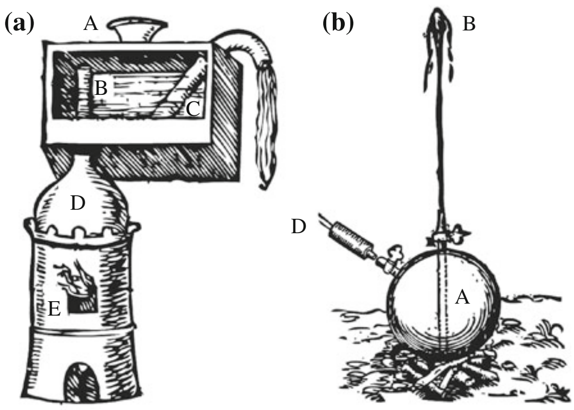

Fig. 7.13 **a** Della Porta's improved Hero's fountain. **b** De Causs' thermal engine (Redrawn from [116, p. 75; 115, p. 4r])

[115], imagined that, see Fig. 7.13b, a fluid entering from D in A is raised to B by the steam generated by a fire heating A.

Otto von Guericke (1602–1686) dealt with the possibility of exploiting the vacuum to produce mechanical motions. He showed that the air pressure can push a piston into a cylinder in which a vacuum has been made and that it is able to produce mechanical work. Huygens and his collaborator Denis Papin (1647–1712) are credited to have experimented with the possibility of an engine in which a piston was moved by the explosion of gunpowder [652]. This is what Huygens wrote in 1686 in a letter to an unknown correspondent:

> I showed Mr. Colbert[176] a machine that I built with this same intention and which was recorded in our Académie, the effect was that a small amount of powder such to fill a thimble sewing, was able to raise some sixteen hundred pounds, at the height of five feet, and this without the usual impetuosity, but with a force tempered and uniform, and four or five lackeys, that Mr. Colbert suspended to the rope attached to this machine were raised very easily into the air. *But it meets some difficulty to continually renew this force* [emphasis added] [189].[177] (A.7.42)

Papin continued these studies by replacing the gunpowder with steam; his results were published in 1690 in the Acta Eruditorum [282].

Figure 7.14 shows the steam machine built by Papin at laboratory scale. A small quantity of water is placed at the bottom of the cylinder A; a fire is lighted beneath it and the steam formed soon raises the piston B to the top, where a latch E, engaging a notch in the piston-rod H, holds it up. The fire being removed the steam condenses, a vacuum forms inside the cylinder and, the latch E being disengaged, the piston is driven down by the atmosphere pressure and may raise a weight which has been attached to D through a rope and pulleys. The machine had a cylinder of about 6.5 cm in diameter and performed a work of about 1 kgm a minute.

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[176] Jean Baptiste Colbert (1619–1683), a minister to the young French King Louis XIV.

[177] vol. IX, p. 79. My translation.

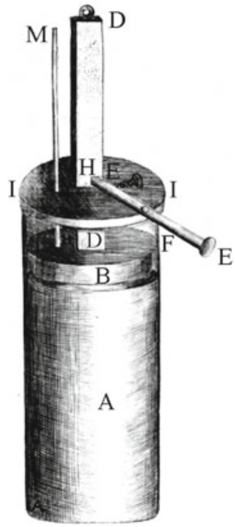

Fig. 7.14 Papin steam engine (Redrawn from [282, Table X, Fig. 1])

Two different views influenced the development in the XVIII century, when steam came to be considered as a possible source of energy alternative to water. On the one hand there was the line from the air pump to the piston-in-cylinder principle in the hands of von Guericke, Hooke, Boyle, Huygens and Papin. On the other end there was the line started by della Porta. It was Thomas Savery (1650–1715) who obtained a patent in 1698 based on della Porta's ideas to drain the water flooding the mines. This machine, however, was not adequate for the purpose for many reasons, one of which was the danger of explosion due to the high pressure necessary for a correct working, and consequently had a scarce diffusion; one of them was used to drain mines while the others were used as pumps to provide drinking water for large buildings, country houses, etc. In the machine of Savery the steam coming from a boiler was sent, via a pipe, inside an ellipsoidal vessel. Subsequently, the vessel was cooled by means of a jet of water from the outside. Following this, the steam present therein condenses causing the vacuum. In this way, the water from the draining mine was aspirated in the empty container. At this point a new jet of steam from the boiler, first makes the water to flow and then fills out the container. The cycle could then be repeated. There were two ellipsoidal vessels which were alternately filled and emptied for greater efficiency. To achieve this it was necessary to open and close alternately some valves; these operations were done manually. The machine had a limit to raise the water not more than about 10 m (Torricelli's limit). Savery solved somehow this limitation by using steam under pressure to push the water inside the container, but with large energy consumption and low efficiency.

Thomas Newcomen (1664–1729) who had studied in Devon with Savery followed a different path, derived directly from the studies of Papin. Newcomen was a man of learning and was in touch with Robert Hooke who probably informed him about

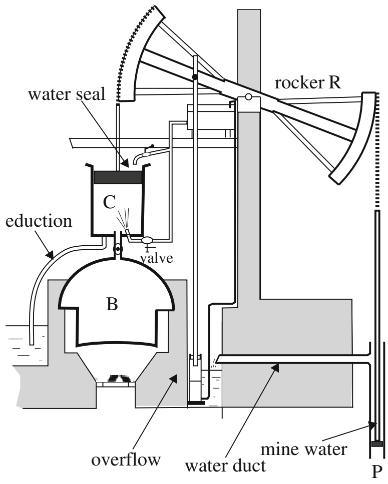

Fig. 7.15 Newcomen's steam machine

Papin's progress [530],[178] coming to the much more efficient machine shown in Fig. 7.15.

Newcomen's machine adopted the cylinder and piston of Papin and worked, unlike that of Savery, at low pressure, that made it easier in construction. It was very reliable for the rather unsatisfactory engineering practices of the time; indeed Newcomen had experience of mines and worked with a skilled plumber. Figure 7.15 shows the basic elements of the machine; a stove fueled a boiler B that produced steam at atmospheric pressure. This steam was released from the bottom into the cylinder C and, aided by the rocker R that kept initially in equilibrium the pump P placed at the opposite end of the rocker arm, made the piston to be lifted.

As soon as the steam had filled the cylinder, cold water was led into it through a valve originating condensation of the steam and a vacuum; soon the piston fell down because of atmospheric pressure. In this phase, the pump was operated for lifting water from the mine. At this point the cycle could start again. Systems of opening and closing of the valves for the entry and discharge of the steam (and water) were automated through the motion of the rod of the injection pump synchronized with the motion of the rocker. The possibility of such automated equipment, not existing in origin, was advised by a young worker assigned to the openings and closings of the valves, Humphrey Potter (fl 1770). The whole apparatus was very large: to give an idea think that the height of the cylinder alone could reach almost 4 m. The rocker realized 12 oscillations per minute in each of which 45 L of water from 46 m depth (through the use of a series of pumps) was raised. Its power could be estimated at around 5 horsepower. This machine, as mentioned, had a great success and over sixty years more than 120 specimens were built.

It was John Smeaton who in 1767 started a scientific study of the steam engine and of what it was able to give, just putting in relation the fuel consumed and the work

[178] p. 384.

produced per unit of time. In such a way it would have been possible to compare the performance of individual machines. Confident of the studies carried out in the laboratory on models (he varied one parameter at a time while keeping the others constant), especially on the water wheels, Smeaton was able to double what we now call the performance of the steam engines. He determined what was the best combination between the diameter of the cylinder, of the piston, the speed of operation, the surface of boiler, the water supply and the coal consumption for a given power output. In carrying out his experiments he saw, not without surprise, that the steam in the cylinder might not be condensed completely for the maximum power of the machine. The complete condensation would have given a substantial boost to the piston but would have slowed the machine since, cooling the cylinder, a greater amount of steam was necessary to be heated starting from the complete cooling. With a residual steam something was lost in the impulse of the plunger but earned more in the speed of the machine. Smeaton also realized that a mixture of steam and air was more efficient because the air, non-condensing, would be naturally arranged in the manner of an insulation collar between the steam and the cold wall of the operating cylinder. Notwithstanding Smeaton's work, the useful mechanical work still remained at the level of 1 % of the heat that had been used.

Great Britain in the last half of the XVIII century marched with continuous increases in production. The energy availability was good but progressively started to dwindle. At least since the English parliament had liberalized the production of cotton fabrics (1774) the water power was no longer enough; moreover it had the great limitation of not being available anywhere. Then the need was felt to have more power sources to locate where it was considered necessary, also in connection to the development of metallurgy. The first steam engines were however too bulky and not adaptable to the variable power in an area generally smaller than that concerning the draining of mines.

The steam engine came to be examined in 1763 by James Watt (1736–1819), a mechanician and manufacturer of precision instruments at the University of Glasgow who occupied a workshop in the same university. Watt's approach was that of a scholar, not initially interested in the production of machines. He was forced to repair, for the University, a small model of Newcomen's engine (a cylinder with a diameter of 5 cm and a height of about 20 cm; a boiler with a diameter of 23 cm).

Watt had understood that what is causing the sudden steam exhaustion arose from the excessive cooling of the metal of the cylinder after placing it in the water at every stroke of the piston. Watt realized that, for a better operation of the machine it would have been necessary that the cylinder was kept at the same temperature as the steam and that the water resulting from condensed vapor returned to a temperature not higher than 37.7 °C (Watt knew that at this temperature, in a vacuum, water starts to boil). To maintain the cylinder at the temperature of the steam he built around it a jacket of wood such that between this and the cylinder itself the steam could circulate (an idea probably taken from Smeaton). In his investigation he also discovered the existence of the latent heat of steam, independently of his friend Joseph Black (1728–1799) who taught chemistry at the same university.

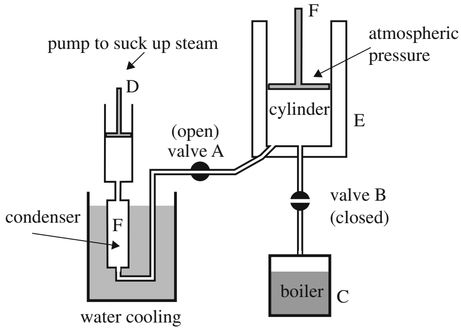

Fig. 7.16 A basic version of Watt's steam engine

Watt wrote that it took him two years to well understand how to operate to accomplish what he had understood: it was necessary to open a communication between the cylinder containing the steam and another vessel in which the vacuum was realized. Thus the steam coming from the cylinder would have immediately penetrated it and would have continued to penetrate until the equilibrium was reached between the cylinder and the new vessel. And if it had been maintained very cold, then as soon as the steam would come in it, it gradually condensed. The idea of a condenser separated from the steam cylinder, was the most important discovery at the purpose of the final success of the steam engine. At this point, one had to go to the practical part: the money to build the first machine. But this is another story, it suffices to say that Watt joined the manufacturer and engineer Matthew Boulton (1728–1809) and in 1788 an already very sophisticated machine began to spread first in Great Britain and then in Europe.

Figure 7.16 shows the basic aspects of Watt's machine whose main elements were the great boiler (C), the cylinder equipped with a steam jacket (E), the separate condenser (F). The steam produced by the boiler enters the cylinder and lifts the piston (in this phase the valve B is open and A is closed). As soon as the piston has reached the top of the cylinder, A opens, B closes and a pump D sucks the steam from the cylinder. The cylinder goes down because of the atmospheric pressure. The sucked steam goes to the condenser to return to the liquid state. The valve B is reopened and the A closed, to begin a new cycle. Watt initially thought to directly connect the appliances to the piston F, but then he used the system of Newcomen's rocker. The rocker also drives the pump D that sucks the steam from the cylinder.

In 1782 Watt built the double effect machine, which basically doubled the power of the simple machine with the same volume. Essentially the steam was introduced alternately on the two faces of the piston. In this way, the direct intervention of the atmospheric pressure to bring down the piston was abandoned and the possibility of machines with cylinder no longer necessarily vertical was opened. The problems with the dual effects were related to the transfer of motion to the rocker. The chain was no longer usable and a rigid mechanism was needed. Watt brilliantly solved this problem also. Finally he realized a centrifugal regulator valve, the so-called Watt pendulum, (added in 1788), a mechanism that regulated the supply of steam in order to keep the machine in motion with constant velocity.

From the above it would seem that the invention of the steam machines has followed a path independent of science and in particular of thermodynamics. This independence of science is usually also seen for other important inventions such as the telephone, the bulb and even the radio. In reality, at least for the steam machines, the involvement of science and scientists was important and without the development science had achieved in the XVIII century there would have been no steam machines. Meanwhile, an examination of the names of the characters that were interested in machines shows that they were largely either scientists or other people of good scientific culture. It must however be noted that notwithstanding many concepts belonging to what today is called thermodynamics were known, both scientists and engineers of the time were not aware of the very 'principles' of thermodynamics, in particular the first principle asserting the equivalence of heat and work and the second principle for which only a part of heat can be converted into work in passing from a higher to a lower temperature. Moreover Sadi Carnot's *Réflexions sur la puissance du feu et sur les machines* of 1824 remained practically unknown, with its important affirmation that the theoretical efficiency of a thermal machine is independent of the medium used, be it steam, air, alcohol. The lack of knowledge of the principles of thermodynamics among engineers lasted at least until 1850 [374].[179] This made the development of the steam engines at high pressure and internal combustion engines not straightforward as it could be and as it then was with the complete acquisition of thermodynamics by the community of engineers.

I will confine myself here to comment on the case of James Watt. For sure there was an indirect influence of science: for example the development of metallurgy made possible the use of various metals, including iron, which are required for the proper functioning of the steam engine. The development of methodology of systematic experimentation was also important. The experimental approach of Watts was that of a scientist. Not being an engineer involved in the construction of full size engines he experimented with laboratory scale models, and so he could proceed with more freedom.

There was a direct influence also, of thermodynamics in particular. Watt maintained contact with science throughout his life. As a member of the Lunar society and a Fellow of the Royal society, he was in touch with leading scientists, among which were Joseph Black (1728–1799) and John Robinson (1739–1805), the most important English scholars of the emerging science of thermodynamics [530]. In the following quotation Watt listed the scientific knowledge which was useful to him:

> Though I have always felt and acknowledged my obligations to him for the information I had received from his conversation, and particularly for the knowledge of the doctrine of Latent Heat, I never did, nor could, consider my improvements as originating in those communications […]
>
> But this theory, though useful in determining the quantity of injection necessary where the quantity of water evaporated by the boiler, and used by the cylinder,was known did not lead to the improvements I afterwards made in the engine. These improvements proceeded upon the old-established fact,that steam was condensed by the contact of cold bodies, and the later known one, that water boiled in vacuo at heats below 1,000, and consequently that a

[179] p. 163.

> vacuum could not be obtained unless the cylinder and its contents were cooled every stroke below that heat. These, and the degree of knowledge I possessed of the elasticities of steam at various heats,were the principle things it was necessary for me to consider in contriving that new engine [530].[180]

Considered from the point of view adopted in this book, the motion of bodies, the invention of the steam engine introduced a great new feature. It in fact widened the Aristotelian category of bodies that move by themselves. Besides the living beings there were thermal machines. No matter if their motion did not come from nothing, the important fact was that a motion without applied external forces was made possible. The analogy between heat engine and live being reached its peak with the construction of the steam locomotive, for the first time a being without a soul moved by itself.

The steam locomotive appeared in the early 1800s in England, as a substitute for horses for towing convoys of trucks of coal mines and soon the potential was clear in spite of the first prototypes having many flaws and steam production being rather poor given that the boilers used, with a vertical structure, were little more than a big pot on the fire. A more efficient boiler, known as the tubular boiler, was developed by Marc Seguin (1786–1875) in 1829. The first steam locomotive to run on rails at the head of a convoy of goods and passengers was the Penydarren by Richard Trevithick (1771–1833) in 1804. However, the first truly effective steam locomotive was the Rocket by George Stephenson (1781–1848) and his son Robert, who proposed, in the 1829 world competition (the Rainhill Trials issued by the Company for Railway Track, from Liverpool to Manchester), an innovative tubular boiler; the Stephensons won the competition. The Rocket was already a modern machine, whose technical solutions would be revised and improved later on subsequent projects, but basically represented the classic steam locomotive.

7.6.3 The Energetism

In the second half of the XIX century the mechanistic vision of the world of physicists entered a deep crisis because of the difficulties of explaining new phenomena, such as the thermal ones, which had been known for a long time, and magnetic and chemicals that were new. In the following I offer a short account of the way some scientists faced the problem; in particular I will discuss positions of Duhem, Poincaré and Mach, then comment on the extremist positions carried forward by Wilhelm Ostwald.

Mach's criticism of the mechanism derived from his positivism; that is from the requirement to avoid assumptions that are not strictly necessary. And the existence of elementary particles (atoms), fundamental for mechanism, was not considered strictly necessary neither to explain the phenomena usually classified as mechanical nor other kinds of phenomena. According to him the mechanistic conception of nature

[180] Letter of James Watt to Dr. Brewster, May 1814, pp. 385–386.

appears as historically justified, perhaps even temporarily useful, but completely artificial.

Mach cited thermal power as an example not necessarily reducible to mechanics. But it is probably with physiology that the mechanistic model shows its full weaknesses.

> It would be equivalent, accordingly, to explaining the more simple and immediate by the more complicated and remote, if we were to attempt to derive sensations from the motions of masses, wholly aside from the consideration that the notions of mechanics are economical implements or expedients perfected to represent mechanical and not physiological or psychological facts. If the means and aims of research were properly distinguished, and our expositions were restricted to the presentation of actual facts, false problems of this kind could not arise [566].[181]

Poincaré did not take a precise position on mechanism. He observed that it exhibits some difficulties to explain many phenomena and even to explain its foundation. If instead of force one assumes energy as leading concepts, things go a little better. A theory based on the concept of energy has the following advantages:

1. It is less incomplete, that is to say, the principles of Hamilton and of the conservation of energy teach us more than the fundamental principles of the classical theory, and exclude certain motions which do not occur in nature and which are compatible with the classical theory.
2. It frees us from the hypothesis of atoms, which was almost impossible to avoid with the classical theory [605].[182] (A.7.43)

But it raises new problems, as for instance the definition of energy.

In simple cases one can easily make recourse to kinetic energy (T), to potential energy (U) and heat also (Q). But in general it is not possible to individuate in a rational way the structure of these three quantities. Poincaré used his conventionalist position and reduced the principle of conservation of energy to the generic and not experimentally verifiable statements:

> There is something that remains constant.

Poincaré at this point stopped, conceding that the argument was culminating into philosophy and was difficult to detangle and observing that in any case the principle of energy conservation can be proved only for reversible processes, but this is not the case in nature.

Duhem developed a unified theory of mechanics:

> The reduction of all physical properties to combinations of figures and movements or, according the used nomenclature, the mechanical explanation of the universe, now seems doomed. It is not condemned by metaphysical or mathematical a priori reasons. It is condemned because it has so far been nothing but a project, a dream and not a reality. Despite tremendous efforts, physicists have never been able to devise an arrangement of geometric figures and local movements, treated according to the rules of rational mechanics, giving a satisfactory representation of a set somewhat extended of physical laws.

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[181] p. 507.

[182] p. 149. My translation.

> The attempt that aims to reduce all Physics to rational Mechanics, which was always a futile attempt in the past, is it intended to pass a day? A prophet alone could answer affirmatively or negatively to this question. Without prejudging the direction of this response, it seems wiser to abandon, at least provisorily, these fruitless efforts toward the mechanical explanation of the Universe. We will thus try to formulate general laws for bodies to which all physical properties must obey, without assuming a priori that these properties are all reducible to geometry and local movement. The core of this general laws no longer will reduce to rational Mechanics [125].[183] (A.7.44)

To indicate the more general science of mechanics Duhem used two terms; one is *thermodynamics*, which is connected historically to such a science; the other is *energetic* due to Rankine which immediately evidence the fundamental concept.

> The code of the general laws of physics is known today under two names, the name of Thermodynamics and the name of Energetic. The name of Thermodynamics is closely connected with the history of this science; its two key principles, Carnot's principle and the principle of conservation of energy, were discovered by studying the motive power of a thermal machine. This name is still justified by the fact that the two notions of work and quantity of heat are constantly involved in the reasoning by which this doctrine develops.
> The name of Energetic is due to Rankine; the energy idea being the first that this doctrine is to define, that which is associated with many other concepts. This name seems no less well suited than Thermodynamics [125].[184] (A.7.45)

7.6.3.1 The Role of Wilhelm Ostwald

The scientists I quoted saw energy as a more suitable and perhaps more interesting concept than that of force of the mechanist point of view, but they did not declaim completely the mechanism. Wilhelm Ostwald (1853–1932) took a quite different position. Notwithstanding that he was a scientist (he won the Nobel prize for chemistry in 1909) and a very appreciated professional chemist, at a certain point he put on the role of philosopher, or better of the prophet of a vision of a world founded on the concept of energy: the energetism which largely remained a German phenomenon. In this he was flanked by Georg Helm (1851–1923).

Of Ostwald's many propagandistic writings, the most famous and maybe most important, were the *Natural philosophy*[185] [277] and *Die energie* [276].

Natural philosophy
One of Ostwald's main concern was to deny a mechanistic vision of the world; for him matter is not simply something inert but is an object of changing, of a 'force' which expresses a continuous evolution [597]. In his *Natural philosophy*, after having presented some general concepts, Ostwald procedeed to examine physics, and mechanics in particular, which according to the 'classical presentations of this science', is divided into statics and dynamics.

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[183] vol. 1, p. 2. My translation.

[184] vol. 1, p. 3. My translation.

[185] The book was first published in German in 1908 as *Grundrißder Naturphilosophie: Bcher der Naturwissenschaft*, but the English edition revised by the author is more known.

In statics the fundamental concept is that of work. Work is a quantity which is conserved; this fact is sufficient for Ostwald to consider it as a substance:

> This discovery, that there is a magnitude which can be quantitatively determined, and which, as experience shows, remains unchanged, however much its factors may change, invariably results not only in a very simple and clear formulation of the corresponding natural law, but also corresponds to the general tendency of the human mind to work out conceptually "the permanent in change." If, in accordance with the word-sense, we denote everything which persists under changing conditions by the name of substance, we encounter in work the first substance of which we have attained knowledge in our scientific journeys. In the history of the evolution of human thought this substance has been preceded by others, especially by the weight and mass of ponderable bodies (which are also subject to a law of conservation), so that at present we are inclined to connect with the word substance a tacit secondary sense of ponderability. But this is a remnant of the still very widely spread mechanistic theory of the universe, which, though it has almost finished its role in physics, will presumably continue to persist for a long time to come in the popularly scientific consciousness in accordance with the laws of collective thought [277].[186]

Passing to dynamics, the concept of work is complemented with that of kinetic energy (Ostwald's term). Work is no longer conserved, but the sum of work and kinetic energy is. That allows one to introduce the concept of energy in a general form, which however maintains the status of substance:

> Thus, while work can be called a substance only in a limited sense, since its conservation is limited only to perfect machines, we may call energy a substance unqualifiedly, since in every instance of which we know the principle has been maintained *that a quantity of any energy never disappears unless an equivalent quantity of another energy arises*. Accordingly, this law of the conservation of energy must be taken as a fundamental law of the physical sciences. But not only do all the phenomena of physics, including chemistry, occur within the limits of the law of conservation, but until the contrary is proved the law of conservation must also be regarded as operative in all the later sciences, that is, in all the activities of organisms, so that all the phenomena of life must also take place within the limits of the law of conservation. This corresponds to the general fact, which I have emphasized a number of times, that all the laws of a former science find application in all the following sciences, since the latter can only contain concepts which by specialization, that is, by the addition of further characteristics, have sprung from the concepts of the former or more general sciences [277].[187]

Ostwald extended his concept of substance-energy to heat, to electricity, to chemistry. The last chapter of *Natural philosophy* was devoted to the biologic and social sciences, trying to individuate the various kinds of energies that intervene.

Die energie

Natural philosophy concerned philosophical matter; *Die energie* had instead a historical point of view. In the following I will give a large extract of this book because it contains the history of thermodynamics as seen by one of the leading actors. The introduction presents the author's intent in a very clear way:

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[186] pp. 131–132.

[187] p. 136.

> The purpose of these pages is to know the history of the development and content of a concept whose beginnings were as small as those of the first seed that the earth brought when the temperature had dropped enough to be compatible with life. This concept took a more varied shape and adapted gradually to the most diverse facts. It conquered one desert after another. Its vitality and adaptability proved to be so great that today we cannot represent our region so arid, to the high where the air is so rarefied that life forms cannot be prosper. We do not expect nothing less than the gradual extension of its rule in all the areas of science. Undoubtedly, its dominance is not of such a nature that no other concept can find a place next to or above it. There are some more abstract than it, and, consequently, higher, in a sense. But we do not know what is at the same time as general and also able to explain the specific facts, so inclusive and also leading to precise statements. We never found a so living incarnation of human knowledge. We cannot cite phenomenon that cannot be attached. Among the many concepts, such as number, time, space, etc., that we are trained to give us a theory of our world, no one can express many things relating to the content of this world, to express such things as precisely or to also connect well together.
> This concept is that of energy. [276].[188] (A.7.46)

This history starts from the very beginning to evidence an embryonic notion of energy, contained in the principle of virtual work, which Ostwald attributed to Aristotle in the form: in a machine there is equilibrium as soon as the virtual works compensate. After having made reference to Leonardo da Vinci, Girolamo Cardano, Guidobaldo dal Monte, Giovanni Battista Benedetti, René Descartes, Ostwald devoted some space to Galileo Galilei and Evangelista Torricelli. It is then the turn of Johann Bernoulli and Louis Joseph Lagrange to whom the modern formulation of the principle of virtual work is attributed.

Space is devoted to the principle of the impossibility of perpetual motion. For Ostwald it was an empirical principle, as all the attempts made to nullify it have failed. However, in some cases, as for example the motion of planets, it has not yet registered a slowdown and then some doubt about the validity of the principle could be raised. But the principle can be reformulated endowing it with a greater empirical certitude, by asserting that the case of a creation of work has never been observed, or better that work has never been created without some other form of energy being subject to change.

Ostwald's thesis is that the principle of virtual work and the principle of the impossibility of perpetual motion comprised a unique principle. This thesis is clearly not historically founded and in fact Stevin refuted the principle of virtual work to accept that of the impossibility of perpetual motion and Lagrange, the champion of the virtual work principle, nowhere declared the impossibility of perpetual motion. It is however true that a suitable formulation of the principle of virtual work can be obtained from the impossibility of perpetual motion.

After having argued on the two principles, Ostwald focused on the concept of work, noting that, at least in statics, it is an invariant; moreover he resumed the thesis of *Natural philosophy* by asserting that it is one of the forms of energy, whose properties were discovered first. Ostwald continued by introducing the second discovered form of energy, that associated to motion, or kinetic energy. If kinetic energy is measured by the square of velocity times mass, as suggested by Leibniz, and halved, one

[188] pp. 3–4. My translation.

has the principle of conservation of mechanical energy. There are however cases in which mechanical energy is not conserved, for example in presence of friction. To overcome this difficulty one has to introduce another form of energy, heat. So the invariant is given by the summation of work, kinetic energy and heat.

The first explanation of the transformation of kinetic energy into heat was that by Leibniz who assumed the macroscopic kinetic energy transferred into the microscopic particle composing the bodies, but Ostwald attacked this position as not scientific and even contrary to the progress of science:

> We know that this is an expedient proposed even before the work of Mayer, expedient indicated by Leibniz, and we are convinced that we cannot make real progress in using it and we are condemned to stagnation. The use of this expedient has so delayed the progress of science, at the point that scientists have only recently reached the point where Mayer was sixty years ago. If today a physicist or a chemist wants to show himself as a progressive man, he says that matter and energy are similar or parallel entities, and defines the physical sciences as the science of the transformation of these two indestructible things, matter and energy, without knowing, most of the time, he is merely reproducing the Mayer's design [276].[189] (A.7.47)

Ostwald's hero was indeed the German physician Julius Robert Mayer (1814–1878). The story of the discoveries of Mayer and the publication of his works are well known; this notwithstanding it is worth following the story by Ostwald. Mayer had a first idea of the convertibility of work into heat based on his experience as a doctor on board a ship sailing toward the Antilles. He observed that the blood of sailors was becoming more red as the temperature increased. Because Lavoisier had proved that human heat derives from nourishment, Mayer thought that a part of the heat may be furnished to the body by the prevailing ambience and the warmer the place the more red the blood as it contains more oxygen that is not used in alimental combustion. From this the idea originated, actually not very obvious, that mechanical work produced by an animal and its heat came both from food; and thus if one worked more there was less heating. In other words, work and heat are interchangeable forms of energy.

Ostwald discussed Mayer's difficulties in publishing his results in the renowned journal Annalen der Physik und Chemie, edited by Johann Christian Poggendorff (1796–1877). Those difficulties were mainly to errors contained in Mayer's memoirs but even by the hostility of many German scientists toward *Naturphilosophie*, to which the metaphysical vision of Mayer could be associated.

Finally in 1842 Mayer was able to publish his paper *Bemerkungen über die Kräfte der unbelebten Natur* which is related to the phenomena of the immaterial world only. In subsequent works Mayer discussed also the problem of life and the cosmos [276].[190] Ostwald reported Mayer's memoir in full in *Die energie*, contributing to its diffusion, made easier by the French translation of his text, *L'énergie* [278]. Given its historical interest a large epitome of Mayer's memoir is reported below:

Considerations on forces of inanimate nature

[189] p. 60. My translation.

[190] p. 58.

Forces are causes: therefore the principle *causa aequat effectum* fully applies to them. If the cause c produces the effect e, then $c = e$, if e in turn causes another effect f, $e = f$, etc. Therefore $c = e = f \cdots = c$. In a chain of causes and effects, a term or part of a term can never, as a result from the nature of an equation, become zero. To this first property of all causes we give the name of *indestructibility*. If the given cause c has produced the effect e that is equal to it, c has for that very reason, ceased to exist, it has became e, if, after the production of e, c would have still remained in whole or in part, the cause remaining should reflect an additional effect, so the effect should be $c > e$, which is contrary to the hypothesis $c = e$. Since then c changes into e, e in f, etc., we must consider these magnitudes as different forms of one and the same object. The ability to take different forms is the second essential property of all causes. In substance one can say: the causes are indestructible (quantitatively) and variables (qualitative) objects.

Nature presents two categories of causes between which experience shows that there is an insurmountable barrier. The first category includes the causes with the properties of being ponderable and impenetrable; they are the matter. The second includes the causes that lack these properties, they are the forces, also called imponderable because of the negative property that characterizes them. Forces are *indestructible objects, variables and imponderable*. A cause which determines the elevation of a weight is a force, its effect, the weight raised, is also a force, thus expressing this in a more general way, one say: *any spatial variation of ponderable objects is a force,* as the force determines the fall of bodies, we call it *force of fall*.

Watching gravity as a force, is to imagine a cause that without diminishing itself produces an effect, and therefore, to represent inaccurately a causal chain of things. For a body to fall, its elevation above the ground is no less necessary than its gravity, we must therefore not to attribute to the gravity only the falling of bodies.

One sees, in countless cases, a motion to stop without producing another motion or raising a weight, but a force cannot vanish, it can only take another form, the question therefore arises to know what other form the force we knew as falling force and as a motion, can take. The experiment alone can tell us in this regard. Since it is clear that in many cases (the exception proves the rule), we cannot find for the motion which disappears other effect than heat, one prefers to assume that the heat comes from the motion, rather than to admit a cause without an effect or an effect without a cause, as well as the chemist, when he sees the hydrogen and oxygen disappear and the water to rise, instead of just see these as two different phenomena, he says there is a link between the disappearance of hydrogen and oxygen and the appearance of the water.

But just as one cannot conclude from the relationship between the force of falling and motion, that the essence of the force of falling is motion, one cannot conclude from the relationship between motion and heat, that the essence of heat is motion. One would conclude the opposite, that is to say, in order to become heat, motion—either a single or a vibratory motion, such as light, radiant heat, etc.—must cease to be the motion. If falling force and motion are equal to the heat, the heat must also naturally be equal to the motion and the force of falling. As well as the effect of heat arises in a decrease in volume and a cessation of motion, as well as heat disappears as cause when it manifest its effects, motion, increased volume, elevation a weight.

Applying to gases the principles just established, one finds that the lowering of a mercury column compressing a gas is equal to the amount of heat generated by compression, and from this its results that a falling weight from a height of 365 m produces an amount of heat able to raise from 0° to 1° the same weight of water [278].[191]

Mayor's main idea was to attribute to energy a seal of reality, a substance like matter but distinguished from it.

> For our general examination the most essential thing Mayer did, is the substantive conception of what he calls force, that is the energy. This it is quite the mechanical equivalent of heat. A reality that a certain character and its own kind, just the indestructibility and uncreability marks his reality. To make to enhance this fact, he *cancels the energy from matter*, so that after him there are from one side the indestructible ponderable objects, as the matter, and on the other hand, the indestructible imponderable objects, as the energies [276].[192] (A.7.48)

After the long exposition of Mayer's idea's Ostwald passed to Joule's and Helmholtz's (and Rankine's) works. They are considered interesting but less original and fundamental than those of Mayer because founded on the harmful hypothesis of the mechanical nature of heat. The law of conservation of energy is qualified as the *first principle of energetics* to distinguish it from the *second principle of energetics*, attributed to the young French engineer Sadi Carnot. Ostwald only made reference to the published work of Carnot *Réflexions sur la puissance du feu et sur les machines* [73]; here heat was considered as a fluid that made work passing from a higher temperature to a lower temperature but conserving itself. A certain space was devoted to Carnot's text also to show that the conservation of heat is not crucial for the validity of his results. Ostwald gave a great tribute to William Thomson and Robert Clausius for having harmonized the first and second principle of energetics, noting that a part of heat passing from higher to lower temperature is actually lost and transformed in work, and introducing the concept of entropy.

Ostwald's book ends with reference to the modern conception, in particular those of Willard Gibbs (1839–1903) and Georg Helm (1851–1923), concluding:

> Each form of energy has the tendency of passing from its present state of higher intensity to a state of lower intensity. It is activated if it can follow this purpose. [276].[193] (A.7.49)

7.7 Final Remarks

The XIX century was a very complex period in the history of mechanics and physics featuring controversies internal to mechanics, e.g. between atomistic and continuum conceptions and between force-like and energy-like conceptions of the causes of motion, and external controversies, between mechanics and other areas of physics

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[191] pp. 64–72. The mechanical equivalent of heat referred here is then $365 \times 9.8 = 3{,}397$ J/Kcal; much lower than the value 4,185 assumed today.

[192] pp. 58–59. My translation.

[193] p. 103. My translation.

including thermodynamics and electromagnetism. These controversies would lead to a major revision, not just of the mechanical theory content but of its internal organization and its role within physics. These contrasts are still actual. Below the various positions are briefly summarized.

7.7.1 Internal Controversies

For the sake of simplicity here only the energy/force controversy is discussed. This opposition stems largely from contingent reasons. Because of contingent reasons, Newton, established the force-like conception. Because of contingent reasons later, especially after the development of thermodynamics, some scientists resorted to an energy approach. Supporters of the two positions criticized each other's basic concepts. The energy-like supporters claimed that force was a confusing and unnecessary concept. The force-like supporters claimed the opposite.

In fact, the majority of scientists were scarcely interested in philosophical discussions and decided to follow one approach over another simply because in their specific area of research it was most heuristically effective. If one wanted to classify scientists by their choice, it could be said that the vast majority of scientists of the XIX century preferred the force-like approach.

Among these scientists there were Saint Venant and Gustav Robert Georg Kirchhoff (1824–1887) who somehow followed D'Alembert's approach, albeit with different shades, for which force was a derived concept, or rather a definition. Kirchhoff expressed his opinions in his *Vorlesungen über mathematische Physik* of 1876 [204]. His exposition has become one of those reported in modern treatises on mechanics (see 1.5.2). Ferdinand Reech (1805–1884) and Jules Andrade (1857–1933) instead followed Euler's approach , for which force must be regarded as a primitive concept, for example, borrowed from statics [306, 6]. A separate mention is deserved by the work of Heinrich Rudolf Hertz (1857–1894) who followed a purely kinematic approach [180]. Hertz, like Kirchhoff and Saint Venant, gave no ontological value to force, but differently from them proposed a mechanical explanation of its apparent reality in terms of hidden masses and motion.

7.7.2 External Controversies

At the end of the XIX century the characteristic dogmas of science: the reducibility of all natural phenomena to the laws of mechanics and the belief that it would eventually reveal the truth of the world, collapsed. With the collapse of mechanism the Aristotelian conception of science that was organized by principles collapsed also. And one began to ask whether any 'true' principle could exist. Here is a quotation from the beginning of the XX century by the philosopher Abel Rey reporting the concerns of the time:

> Today it seems that the framework offered by the physical sciences has completely changed: the general unity gave way to an extreme diversity, not only in the details but in fundamental concepts.
> […]
> The critique of the traditional mechanistic physics that was formulated in the second half of the XIX century, weakened the belief in ontological reality of mechanical physics. The criticism determined a philosophy of physics that became official almost towards the end of the XIX century. According to this philosophy, the science was nothing but a symbolic diagram, a reference system […]. A science that has become simply a pure device, a useful technique, and has no more right to call iself science without the meaning of the word is altered.
> […]
> The failure of the traditional mechanistic science […] involves the proposition: *the science itself has failed* [emphasis added] [611].[194] (A.7.50)

During this time many scientists began to seriously deal with epistemologic aspects of science. The reflections of Poincaré, Mach and Duhem deserve special attention [589]. For the sake of space I will herein refer only to Duhem's attempts to formulate a new mechanical theory, enlarged to account for other phenomena such as heat, electricity, magnetism.

Traditional mechanics qualified by Duhem as *ancient mechanics*, has as its fundamental assumption that every physical system may be reduced to a system of mass points or bodies moving according to the variational principle of Hamilton. The *new mechanics* no longer accepts a vision so simple; it does not hesitate to admit between its equations different terms, such as viscosity, friction, electrokinetic energy [453].[195] According to Duhem it would in principle be possible to maintain the structure of ancient mechanics in any case and at the same time save the phenomena; for example introducing hidden motions and masses, but doing so may complicate dramatically, the mathematics of the problem. An exemplary case of maintenance of the structure of ancient mechanics in physics is that of the English physicists, such as Thomson and Maxwell for instance. They considered their task to find a mechanical image for their equations. But they did not pretend that this image was a true representation of the world, but only a *model*, useful to be used to deduce, analogically, some phenomena that would otherwise be difficult to obtain directly from their equations.

New mechanics based on thermodynamics, tolerates the presence of other principles as well as the classic ones; in this way even though it logically becomes more complex, the application of the theory to concrete cases are much simpler. The creation of mechanics based on thermodynamics is a reaction against the atomistic and mechanistic ideas, a throwback to the deepest principles of the Aristotelian doctrine. The mechanists had wanted to ban physical qualities so they could use a geometric approach. Modern mechanics thinks in terms of quality (heat, viscosity, electric charge), but to do this, it accurately represents them with symbols that can assume numeric values. It is daughter of Aristotle because it is based on quality, it is also

[194] pp. 16–17. My translation.
[195] p. 342.

the daughter of Galileo and Descartes because it is based on a universal mathematics [453].[196]

Duhem was not limited to formulating the general lines of the new mechanics; he indeed gave a detailed formulation based on the rewriting of ancient mechanics using characteristic thermodynamic terms such as *work* and *heat*. His basic concept was that of *activity* (oeuvre) whose definition is quite complex. Duhem formulates this 'definition' as follows:

> Thus, when a system is transformed in the presence of external bodies, we admit that these external bodies contribute to the transformation, either by causing it or facilitating it or blocking it, and this contribution we call the *activity* in the transformation of the system, by the bodies outside that system [125].[197] (A.7.51)

Activity has not necessarily a mechanical nature, e.g. it may consist in the administration of electrical current. However Duhem himself recognized that his definition was "too obscure, vague and mostly impregnated with anthropomorphism" [125].[198] To eliminate these defects he declared that the activity should be simply considered as a scalar physical quantity to be represented with an appropriate algebraic symbol to perform calculations. The whole theory is defined by means of axioms to which it was not attributed a priori a correspondence with the objects of the physical world. From this point of view Duhem's new mechanics can be considered as one of the first instance of a physical theory based on an abstract kind of axiomatization (for the meaning of *abstract* see 1.5.3).

Have once defined the requirements which activity must satisfy, Duhem introduced the concept of total energy. If $G(e_0, \mu_0, e, \mu)$ is the activity made on a system to pass from one arbitrarily chosen reference e_0 and global velocity μ_0 to the generic state e and global velocity μ, it is called *total energy* of the system in the state e and global velocity μ, the expression:

$$E(e, \mu) = G(e_0, \mu_0, e, \mu). \tag{7.57}$$

He went ahead by saying that the following relation holds true:

$$G(e_1, \mu_1; e_2, \mu_2) = E(e_2, \mu_2) - E(e_1, \mu_1); \tag{7.58}$$

that is the activity to go from (e_1, μ_1) to (e_2, μ_2) is equal to the difference between the total energies.

At this point Duhem could formulate a principle of conservation of energy, according to which when any system, isolated in space, undergoes any real variation, the total energy of the system maintains an invariable value [125].[199]

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[196] pp. 343–344.

[197] vol. 1, p. 81. My translation.

[198] vol. 1, p. 81.

[199] vol.1, p. 93.

The total energy principle thus formulated seemed too general to Duhem who restricted it by assuming that the total energy is composed of two terms [125]:[200]

$$E(e, \mu) = U(e) + K(\lambda). \tag{7.59}$$

The first term which depends only on the state e, is named *internal energy*; the second which depends on the local velocity λ (understood in the classical sense) of all system components, is named *kinetic energy*.

In absence of thermal phenomena Duhem established the following fundamental principle (which for him was a theorem), corresponding more or less to the Lagrangian principle of virtual work [125]:[201]

$$\mathscr{T} + \tau - \delta U = 0, \tag{7.60}$$

where $\mathscr{T}$ is the virtual work made by the external forces, τ the virtual work made by the forces of inertia in infinitesimal virtual displacements which meet the constraint condition (bilateral for simplicity).

In the presence of thermal phenomena the principle (7.60) is replaced by the analogous relation [125]:[202]

$$\mathscr{T} + \tau = \delta \mathscr{F}(e, \theta) - \frac{\partial}{\partial \theta} \mathscr{F}(e, \theta)\, \delta \theta, \tag{7.61}$$

where $\mathscr{F}$ is a function of the state variables e which have no longer geometric nature only, but comprehend also the electric charge, the concentration of a chemical component, etc. and moreover depends on the temperature θ. Duhem called this function *internal thermodynamical potential* (potential thermodynamique interne). In isothermal processes it can be replaced by the internal energy U.

[200] vol. 1, pp. 97–98.

[201] vol. 1, p. 185.

[202] vol. 2, p. 3.